Optical Measurement of Wide Bandwidth Ultrasound Fields

by

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To my dear wife Emily,
   and our children:
   Matthew, Christopher, Erin, and Emily
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Hypothesis

Optical Measurement of Wide Bandwidth Ultrasound Fields

by

Todd Alan Pitts

Detailed, accurate, three-dimensional, optical measurements of an instantaneous refractive index perturbation in an optically transparent medium may be obtained from measurements of scattered optical intensity alone. The Method of Generalized Projections (MGP) allows incorporation of these measurements into an iterative algorithm for computing the optical phase as the solution of a fixed point equation. The complex optical field amplitude, computed in this manner, is unique up to a constant unit magnitude complex coefficient. The three-dimensional refractive index distribution may be computed via the Fourier Slice Reconstruction (FSR) algorithm from the optical phase data under the assumption of weak optical scattering. If the refractive index perturbation is induced by an acoustic field, then acoustic pressure, Poynting vector, and particle velocity fields can be computed from these measurements. These field quantities may be obtained at any point throughout the temporal evolution of the sound field, either under mild assumptions on its angular plane wave spectrum or via a minimal number of measurements. Interaction of a wide bandwidth ultrasound pulse with various materials, including biological tissues, may be effectively studied via this measurement method.
Abstract

Optical Measurement of Wide Bandwidth Ultrasound Fields

by

Todd Alan Pitts

Mayo Graduate School

Dr. James F. Greenleaf, Chair

This thesis reports an optical method for detailed measurement of the instantaneous three-dimensional pressure in a wide bandwidth ultrasound pulse. In the basic measurement experiment a collimated laser pulse impinges on the ultrasound pulse and the intensity of the forward-scattered optical field is imaged onto a Charge Coupled Device (CCD) array with a two-lens imaging system and stored. The distance traveled by the scattered optical pulse after exiting the sound field to the plane over which its intensity is recorded is computed from known parameters of the optical imaging system. Delay between ultrasound transducer and laser firing is controlled precisely, allowing the acousto-optic interaction to occur at a specified point in the temporal evolution of the sound field. Immediately after passage through the ultrasound pulse, the optical phase is approximately proportional to the ray sum of instantaneous pressure in the pulse (taken in the direction of optical propagation). The Method of Generalized Projections (MGP) computes this optical phase function from the recorded intensity and known experimental parameters under a scalar Fresnel model for optical wave propagation. A proof of uniqueness (up to a constant unit magnitude complex coefficient) of the computed phase is given. The experiment is repeated at several angles as the ultrasound transducer is rotated through an angle of 180 degrees about an axis normal to the optical axis of the imaging system. The Fourier Slice Reconstruction (FSR) algorithm computes pressure at all points within the ultrasound pulse from the recovered optical phase data. The necessary spatial sampling rate for the optical intensity distributions is approximately four times the highest spatial frequency in the ultrasound pulse. The angular sampling rate required for accurate reconstruction is twice the frequency of the highest angular harmonic in a Fourier series expansion of the spatial frequency domain representation of the ultrasound pulse. The nullspace of the angular sampling operator is explained and its significance for the common sinusoidal structures present in acoustic fields is discussed. A numerical simulation of the complete measurement experiment and reconstruction process is given using a signal peak to noise variance ratio of $8.6 \times 10^{-6}$ estimated from experimental data.

The assumption that the wide bandwidth angular plane wave spectrum of the ultrasound pulse consists only of forward-propagating waves (wave vector components normal to the transducer surface are greater than zero) provides a means of propagating the measurement forward or backward in time under a source-free model. This assumption allows computation of the instantaneous three-dimensional particle velocity and Poynting vector fields from experimental data. It is shown that the recovered phase distribution satisfies a two-dimensional wave equation and may be propagated independently forward or backward in time under a source-free model. A sampling theorem is derived stating that two complete measurements of the three-dimensional pressure field separated in time by $\Delta t$ allow release of the forward-propagating assumption for every acoustic wave number $k$ satisfying $k \neq n\pi/(c\Delta t)$, where $c$ is the acoustic wave speed in the medium and $n$ an integer greater than zero. This provides a means of making detailed measurements of very general ultrasound fields.

A model of the imaging system relating lens location, image plane (optical intensity recording plane) location, and magnification is derived. A calibration experiment agreeing well with the system model is described. Artifacts resulting from dust particles, optical element defects, optical beam nonuniformity, multiple reflections in the optical imaging path, and similar deterministic sources of error that vary slowly with tomographic view angle are considered. A wavelet filtering technique to reduce these artifacts is explained. Comparison of data from a 12.5 mm diameter, 2 MHz, unfocused...
transducer reconstructed with and without filtering indicates very good filter performance.

Complete three-dimensional reconstructions of pressure in a wide bandwidth field from the following sources are presented: annular, semi-circular, and circular apertures using an unfocused 2.25 MHz transducer, elements 32 and 34 of a 64 element, 2.5 MHz linear array transducer, and elements 44 and 21 of the same. Full measurements of a wide bandwidth field from a 12.5 mm diameter, 2.25 MHz, unfocused transducer with and without a cube of beef (approximately 150 mm × 150 mm × 150 mm in size) in front of the transducer are presented and discussed.

A 500 μm diameter needle hydrophone was used to measure three separate pressure time series in a plane approximately parallel to the surface of a 64 element, 2.5 MHz linear array transducer for a wide bandwidth drive signal on the following element pairs: elements 30 and 34, elements 31 and 34, elements 32 and 34. A method of propagating these time series to an arbitrary plane is detailed. The time series for each of the above experiments, propagated to the transducer plane, agrees very well with the location and shape of the driven transducer elements. This method is used to measure the three-dimensional spatial distribution of pressure in the field produced by elements 32 and 34 of a 64 element, 2.5 MHz, linear array transducer. Comparison with optical measurements of the same field made via the optical method described above shows excellent agreement. The greatest disparity occurs in regions of the field where the directivity pattern of the hydrophone prevents accurate measurement (those containing wavefronts impinging on the measurement surface of the hydrophone with significant incidence angles).

Propagation sequences, as well as particle velocity and Poynting vector fields for tomographic views of wide bandwidth fields are presented and discussed for the following: a semi-circular aperture driven with an unfocused 12.5 mm diameter, 2.25 MHz transducer, elements 32 and 34 of a 64 element, 2.5 MHz linear array transducer, elements 44 and 21 of the aforementioned linear array, and elements 27, 31, 34, 36, and 38, together with a 180 degree phase shifted version of the same signal on elements 28, 30, 32, 33, 35, and 37 of the aforementioned linear array. The latter produces a complicated field pattern useful in evaluating the optical measurement method. Propagation of a measured projection of this field back to the transducer emitting surface agrees very well with the known drive pattern.
CHAPTER 1

INTRODUCTION

Modeling and measurement of acoustic wave propagation in various media have long histories in both pure and applied research. Wave propagation plays a critical role in a plethora of medical, industrial, and military applications. Medical imaging involving acoustic fields is widely used to assist in the accurate and early diagnosis of disease, detect structural and functional anatomical or physiological problems, provide feedback during invasive procedures, evaluate the efficacy of a course of treatment, and to increase our understanding of the complex systems and relationships in the human body. Medically relevant imaging extends to areas such as biology, physiology, chemistry, and physics.

The structure of the field used to probe the region of interest is a critical factor in acoustic imaging. Field structure must be understood to obtain an accurate image. It essentially defines the best possible spatial and temporal resolution, determining the finest structures that can be examined and the narrowest velocity differences that can be resolved. Imaging beam side-lobe structure significantly affects the level of noise or clutter in an image. The three-dimensional distribution of power in the beam is very important. In a clinical application it helps determine where a beam is in the balance between increasing signal to noise ratio and affecting or damaging local tissue. The fields produced by acoustic transducers are measured to understand wave propagation, to determine if a beam meets design specifications, to understand or model the interaction of acoustic energy with different structures and materials, as part of the imaging process itself, and for quality control to ensure that a transducer meets safety and quality standards.

This thesis describes an optical method for detailed three-dimensional measurement of instantaneous pressure in a wide bandwidth ultrasound pulse. Sections 1.1 and 1.2 provide relevant historical background and context for the contributions of this thesis to the areas of optical measurement of acoustic fields and the signal processing subject of phase recovery. Section 2.1 describes the basic measurement experiment. The method records the forward-scattered optical intensity from the interaction of a collimated laser pulse with the ultrasound field via a two-lens imaging system. The optical intensity is recorded at several angles as the ultrasound transducer is rotated through 180 degrees about an axis normal to the propagation direction of the incident optical pulse (see Figures 2.1, 4.1, and A.1). The phase of the optical field immediately after passage through the ultrasound pulse is computed from this intensity (See Sec. 3.3). The reconstruction of the instantaneous three-dimensional pressure distribution in the ultrasound pulse is then obtained from the recovered optical phase at each angle (see Secs. 2.1 and 3.4). The scalar Fresnel model used for optical field propagation and the implications of this approximation for this measurement are explained in Sec. 2.2. Section 2.3 describes the linear model used for the relationship between local pressure and local refractive index. A simple delay model for the optical scattering is given in Sec. 2.4. This model is important as it defines the choice of algorithms used to reconstruct the pressure field from optical phase measurements. An
CHAPTER 1. INTRODUCTION

approximate relationship between the temporal and spatial bandwidth of the ultrasound transducer and the spatial frequencies present in the optical intensity pattern is developed in Sec. 2.5 and used in the discussion on sampling considerations for Fourier Slice Reconstruction (FSR) of the ultrasound pulse in Sec. 3.4. A rigorous model for acoustic wave propagation is detailed in Sec. 2.6 allowing the precise definition of pressure and the explicit discussion of assumptions made in the linear model. The linear model for wave propagation given in Sec. 2.6 is used in the remainder of this thesis.

Section 2.7.1 explains that a single measurement of instantaneous pressure under the assumption that the wide bandwidth angular plane wave spectrum of the ultrasound pulse consists only of forward-propagating waves (wave vector components normal to the transducer emitting surface are greater than zero) can be propagated forward or backward in time under a source-free model. In Sec. 2.7.2 a new sampling theorem is derived stating that two complete measurements of the three-dimensional pressure field separated in time by $\Delta t$ allow relaxation of the forward-propagating assumption for every acoustic wave number $k$ satisfying $k \neq n\pi / (c\Delta t)$, where $c$ is the acoustic wave speed in the medium and $n$ an integer greater than zero. Thus, the temporal evolution of very general ultrasound fields may be computed from a minimal number of measurements. Section 2.8 describes how the pressure time series over a plane may be propagated to any other parallel plane. This result is very useful for making hydrophone measurements of a field. It is demonstrated in Chapter 7 and used to compute the three-dimensional pressure distribution in an ultrasound pulse for comparison with an optical measurement of the same field. Sections 2.9 and 2.10 discuss the computation of Poynting vector and particle velocity fields, respectively.

Chapter 3 discusses signal processing aspects of the acoustic field measurement. Section 3.1 explains the relationship between the physical optical intensity and the digitized signal. Section 3.2 describes a two step method for reducing image background corruption in the optical intensity recording resulting from dust particles in the beam path, optical element defects, optical beam nonuniformity, multiple reflections in the optical imaging path and similar deterministic sources of error that vary slowly as the ultrasound field is rotated during the data collection process. As mentioned previously, pressure is reconstructed from optical phase. Optical phase is computed from optical intensity as the solution of a fixed point equation via the Method of Generalized Projections (MGP) (see Sec. 3.3.2). Only a very small class of signals satisfies the fixed point equation, making the computed phase function unique up to a constant, unit magnitude, complex coefficient (see Sec. 3.3.1). Section 3.4 discusses the Fourier Slice Reconstruction (FSR) algorithm, the spatial sampling rate for the optical intensity images, and the angular sampling rate necessary for accurate reconstruction of pressure in the ultrasound pulse.

Chapter 4 describes simulation of the entire experiment. Generation and propagation of the ultrasound field are computed via a Fresnel model for acoustic wave propagation. A scalar Fresnel model for optical wave propagation (see Sec. 2.2.5) and a simple delay model for optical scattering (see Sec. 2.4) are used to compute the forward-scattered optical field at each of several angles as the ultrasound pulse is rotated about an axis normal to the propagation direction of the incident optical wave. Gaussian white noise is added to the simulated intensity images to yield a signal peak to noise variance ratio of $8.6 \times 10^{-6}$ (estimated from experimental data). The Method of Generalized Projections (MGP) (Sec. 3.3.2) is used to compute the phase of the optical wave immediately after passage through the acoustic field, and the Fourier Slice Reconstruction (FSR) algorithm (see Sec. 3.4) is used to reconstruct the numerical ultrasound pulse phantom. The reconstructed pressure field compares very well with the original numerical phantom.

Calibration of the physical system and investigation of the linearity of the instrumentation used to record optical intensity is covered in Sec. 5.1. Linearity of the measurement process itself is examined in Secs. 5.2.1 and 5.2.2 by explicit experimental testing of the superposition and scaling properties satisfied by linear systems. Sections 5.3.1 and 5.3.2 demonstrate the phase retrieval and reconstruction process using experimental data. The latter examines the effectiveness of a wavelet based filter (see Sec. 3.2.2) in removing small scale background corruption.

Chapter 6 gives complete reconstructions of the instantaneous three-dimensional pressure distributions in several different fields (also see Appendix A). Sections 6.1 and 6.2 respectively cover fields from annular and semi-circular apertures driven with an unfocused 2.25 MHz transducer. Sections 6.3 and 6.4 respectively discuss reconstruction of a wide bandwidth field produced with elements 32 and
34 of a 64 element, 2.5 MHz linear array transducer, and elements 44 and 21 of the same. Full measurements of a wide bandwidth field from a 12.5 mm diameter, 2.25 MHz, unfocused transducer with and without a cube of beef (approximately 150 mm × 150 mm × 150 mm in size) in front of the transducer are presented and discussed in Sec. 6.5. The latter experiment demonstrates the ability of the optical measurement method presented in this thesis to quantitatively study ultrasound beam interaction with tissue.

Section 7 presents hydrophone measurements of three separate wide bandwidth fields produced using elements 30 and 34, elements 31 and 34, and elements 32 and 34 of a 64 element, 2.5 MHz linear array transducer. A 500 µm diameter needle hydrophone was used to measure the pressure time series in a plane approximately parallel to the surface of the transducer. The theory in Sec. 2.8 is used to propagate these pressure time series to the surface of the transducer. The shape and dimensions of the fields in the plane of the transducer emitting surface match the drive signals and known source parameters well. The same method is used to compute the instantaneous three-dimensional pressure distribution for the experiment using elements 32 and 34 of the 64 element, 2.5 MHz linear array transducer. This field was measured at approximately the same delay via the optical method presented in this thesis. Hydrophone and optical measurements are registered by hand. Line plots comparing pressure measurements obtained via both methods are in excellent agreement (see Figures 7.5 and 7.6). The greatest disparity occurs in regions of the field where the directivity pattern of the hydrophone complicates accurate measurement (those containing wavefronts impinging on the measurement surface of the hydrophone with significant incidence angles).

Chapter 8 contains propagation sequences and particle velocity and Poynting vector fields for tomographic views (see Appendix B) of wide bandwidth fields from the following sources: a semicircular aperture driven with an unfocused, 12.5 mm diameter, 2.25 MHz transducer, elements 32 and 34 of a 64 element, 2.5 MHz linear array transducer, elements 44 and 21 of the aforementioned linear array, and elements 27, 31, 34, 36, and 38, together with a 180 degree phase shifted version of the same signal on elements 28, 30, 32, 33, 35, and 37 of the aforementioned linear array.

Appendix B shows that the Radon transform of the solution to an N-dimensional, linear, shift-invariant operator equation solves an (N − 1)-dimensional equation. This allows parallel computation of the solution to such problems. This result is used in this thesis for the computation of the numerical ultrasound field phantom used in the simulation discussed in Chapter 4. Appendix C derives a wave optics model for the optical measurement system using an operator notation defined in Appendix D.

Contributions. Sections 1.1 and 1.2 provide context for the following contributions of this thesis to the areas of optical mensuration of acoustic fields, optical tomography, and the image processing subject of phase recovery:

- A new method for fully three-dimensional, non-perturbative, high-resolution optical measurement of instantaneous pressure in wide bandwidth ultrasound fields of arbitrary structure is developed (see Chapters 2 and 3). Several different fields with complex structures are measured optically (see Chapter 6) and are found to match hydrophone measurements very well (see Chapter 7).

- A three-dimensional measurement of tissue-induced phase aberration in a wide bandwidth ultrasound pulse is obtained via the optical method presented in this thesis. Comparison of this measurement with an unaberrated pulse under otherwise identical experimental conditions is presented (see Sec. 6.5), demonstrating a means of making detailed studies of tissue-beam interaction.

- Computation of the full four-dimensional space-time dependence of an acoustic field from a single three-dimensional measurement under a forward-propagating assumption on its wide bandwidth angular plane wave spectrum is reviewed (see Sec. 2.7.1). Computations of particle velocity and Poynting vector fields are also explained (see Secs. 2.9 and 2.10 respectively). The theory is applied to several experimental data sets. Acoustic energy passing through the plane of the transducer emitting surface outside the known aperture indicates the possibility of improving the accuracy of this new optical method through application of a model-based filter (see Chapter 8).
A new theorem is derived, stating that two complete measurements of the three-dimensional pressure field, separated in time by $\Delta t$, allow release of the forward-propagating assumption for every acoustic wavenumber $k$, satisfying $k \neq n\pi/(c\Delta t)$, where $c$ is the acoustic wave speed in the medium and $n$ an integer greater than zero. Thus, the temporal evolution of very general ultrasound fields may be computed from a minimal number of measurements (see Sec. 2.7.2).

A result in the literature [1] on the uniqueness of a real, finite-support, multi-dimensional signal given the magnitude of its Fourier transform is extended to cover complex signals. The extended theorem states that a complex signal is determined uniquely up to a shift, a unit magnitude complex coefficient, and conjugate reflection through the origin by the magnitude of its Fourier transform, if its $z$-transform possesses at most a single non-conjugate symmetric factor (see Sec. 3.3.1).

Application of the Method of Generalized Projections (MGP) to compute optical phase for use in tomographic reconstruction of the ultrasound field requires understanding of the uniqueness of optical phase given intensity in the measurement experiment (see Figures 2.1, 4.1, and A.1). In Sec. 3.3.1 a complex, finite-support, multi-dimensional signal is shown to be determined uniquely up to a unit magnitude complex coefficient, if the $z$-transform of the product of the signal with a multi-dimensional chirp possesses at most a single non-conjugate symmetric factor.

A simple wavelet-based filtering method for handling background corruption due to non-ideal characteristics of the optical beam source, dust particles, optical element flaws, and optical multipath is developed and found to work well (see Secs. 3.2 and 3.2.2).

### 1.1 Optical Measurement of Acoustic Fields

C.V. Raman and N.S. Nagenda Nath provided the first accurate theoretical descriptions of the scattering of light by ultrasound in a series of five papers published in 1935–36 [2–6]. The theory outlined in these papers formed the basis for further investigations into description of the physics and its use for measurement for the next several decades. The original paper, titled *The Diffraction of Light by High Frequency Sound Waves: PART I* [2], considers an experiment in which an optical plane wave impinges normally on a rectangular column of monochromatic ultrasound. In this paper the tone burst is modeled as a stationary phase grating (the amplitude of the optical wave is assumed to remain unperturbed on passage through the ultrasound beam) placing restrictions on the amplitude of the acoustic field and neglecting Doppler shift of the optical wave. Their result describes the Fraunhofer or far-field optical diffraction pattern observed. The second paper in the series [3] extends the above description to include oblique optical wave incidence angles. The next paper [4] describes Doppler shifts in the optical wave induced by the temporally varying nature of the ultrasound field for the case of normal optical wave incidence. Both propagating and standing acoustic waves cases are described. Part IV extends Part III to a general periodic ultrasound wave (although the rectangular nature of the beam transverse to the direction of acoustic propagation is retained) [5]. The final paper in the series discusses some of the implications of optical amplitude changes and oblique incidence on the generalized theory described in Part IV [6].

Further investigations extended this model to include finite extent sound fields, higher ultrasound intensities [7–9], and fields exhibiting various standing wave ratios [10]. Although Raman and Nath include general spatially periodic fields in their original description, most theory and measurement research involve the special case of monochromatic ultrasound fields. The Raman-Nath description and the majority of other investigations are far-field descriptions of the optical scattering in terms of gratings and diffraction orders [11]. However, some research has been done on descriptions of the optical near field [12–14].

The theoretical model of the scattering experiment formed the basis for various measurement techniques. Some investigators observed the optical intensity in various diffraction orders in the far field [15]. Others introduce optical processing elements into the imaging chain, the most predominant of which is a spatial filter in the Fourier plane of a lens (this plane would be equivalent to observation of the far field or Fraunhofer optical intensity) [16–18]. This filter limited the diffraction orders passed
to the first and zeroth allowing observation of their interference pattern. Ultrasound beam parameters are calculated from a knowledge of the structure of this interference pattern and its measured value. Other processing elements include a conventional schlieren knife-edge technique which blocks half of the Fourier plane of the lens, and a dot stop experiment which passes all spatial frequency components except the average value. Unifying theories have been presented explaining the relationship between Raman-Nath models and those valid for other experimental parameter ranges such as higher intensity ultrasound fields [19], higher ultrasound frequencies, and wider beam widths [20]. Later, the integrative nature of the Raman-Nath type models was exploited and tomographic techniques were used to reconstruct pressure profiles for continuous wave ultrasound beams [21–33].

It is important to realize that these models and measurement techniques are parametric in nature. That is, they assume a particular model for the object to be measured. The inversion of the scattering problem then becomes one of parameter estimation. An advantage of parametric estimation is robustness in the presence of noise. A clear disadvantage is the necessity to reasonably accurately model the basic structure of the object to be measured. Many of the investigations measure some aspect of the beam such as power or shape [34]. Others measure a parameter of the material through which it propagates such as absorption, wave speed, elastic constants [35], nonlinearity [36] or even acoustically induced optical anisotropy [37]. Some report measurements of pressure (and in the case of tomographic techniques, cross-sectional pressure), however, these methods generally rely on a rather restrictive model for the sound field.

Research on other optical methods of acoustic field measurement has been reported, particularly in air [38–45]. Among these methods are interferometric and holographic techniques [22,46]. The technique described in this paper makes only a very moderate assumption on the nature of the sound field. Specifically, the sound field must be sufficiently weak so as to allow a simple delay model for the optical scattering. This is critical only because of the tomographic reconstruction technique chosen. It is possible to relax this restriction and spend more effort on the tomographic inversion. The sound field does not have to be periodic in space or time. This technique also produces a complete description of instantaneous pressure in the ultrasound pulse. Furthermore, under very mild assumptions the pulse may be propagated forward and backward in space and time. Particle velocity and Poynting vector fields may be computed as well. Making two complete measurements of the three-dimensional pressure field separated in time by \( \Delta t \) allows release of the forward-propagating assumption for every acoustic wave number \( k \) satisfying \( k \neq n\pi/(c\Delta t) \), where \( c \) is the acoustic wave speed in the medium and \( n \) an integer greater than zero. In this case the four-dimensional structure an arbitrary acoustic disturbance may be reconstructed. Also, the signal quality may be significantly improved (and the system cost lowered) for some applications by the elimination of optical components.

Measurement of the sound field is important in and of itself. Imaging and testing for medical diagnostic and industrial purposes depend critically on the structure of the interrogating acoustic radiation. However, measurement or even mere visualization (qualitative assessment) of the acoustic field can be extremely valuable in evaluating the interaction of acoustic fields with various materials and structures. Many reports in the literature describe the use of optical techniques for the visualization of scattering experiments [25,31,43,47–130]. Very few involve actual measurement of the acoustic field. A very small category of experiments making use of optical techniques for visualization and measurement of acoustic fields may be found in the medical area [131–145]. None of these involve complete measurement of an arbitrary pressure field. This thesis demonstrates the ability of a new optical technique to provide measurements for the study of the interaction between ultrasound and tissue. The complete field from an unfocused transducer is measured with and without the presence of a tissue scatterer in front of the aperture. Reconstruction of the entire three-dimensional pressure field is performed and the effects of the nonuniform acoustic wave speed distribution in the tissue can be readily seen and quantified (see Sec. 6.5).

### 1.2 Signal Phase Recovery

The method for ultrasound field measurement described in this paper is the inversion of an optical scattering problem. Such inverse problems require knowledge of (or some reasonable assumption about) the phase of the scattered optical wave. The importance of optical phase measurement has
led to a vast literature on the subject. Many novel experimental techniques have been developed to record optical phase. In some situations direct measurement of phase is not possible or the quality of the phase measurement is necessarily poor. In these cases it is sometimes possible to determine the phase from a measurement of magnitude and some general constraints on the signal.

Phase retrieval is a subset of the rather large class of problems known as signal recovery. It has applications ranging from astronomy, to remote-sensing, certain types of tomography, holography, electron microscopy, and crystallography. All these imaging or inverse problems use mechanical or electromagnetic wave fields to probe their respective targets. These wave fields are multi-dimensional signals which may be decomposed into Fourier Components, each sinusoidal component having an amplitude (modulus) and a delay with respect to some reference point (phase). In many applications it is possible to measure the Fourier transform of the desired signal. Most often the modulus of this signal is more easily, or more accurately measured than the phase. In many cases the phase is simply not available. Hence we find much discussion in the literature of what is called the Fourier phase retrieval problem. Specifically, given the modulus of the Fourier transform of a signal what is known about the signal itself? Under what conditions on the signal has it been restricted sufficiently to provide useful information? Given such conditions, is there an algorithm that will recover the phase and hence the signal? How will the algorithm perform in the face of noise?

In general, both the amplitude and the phase of the signal are required to specify it uniquely. The ambiguity resulting from loss of the phase information may be described in terms of the z-transform of the signal [1]. Knowledge of the Fourier modulus implies knowledge of the autocorrelation function of the signal from which we may compute the square of the modulus of the signal’s z-transform. This may always be factored into a conjugate pair that is unique up to factors of zero degree. If the original signal is real, inverse z-transform is a conjugate pair that is unique up to factors of zero degree. Two real signals are referred to as equivalent if they are related by a shift, a sign flip, and possible reflection through the origin. If further reduction of the z-transform modulo into a product of $N > 2$ nontrivial (degree greater than zero) factors is possible, reconstruction of an equivalent signal requires choosing the correct $N/2$ factors and computing the inverse z-transform of their product. Determining the factorability of the z-transform of a signal is in general a difficult problem and much research has been done on this topic [146, 147]. Typically, the region of support for the autocorrelation of the signal is used in the determination of z-transform factorability. It has been shown that factorable polynomials with complex coefficients form a set of Lebesgue measure zero [148]. Also, the geometrical structure of the set of factorable polynomials with real coefficients has been shown to be such that an irreducible polynomial generally remains such under small perturbations of its coefficients $[149, 150]$. The general success of phase retrieval in this research is taken to mean that a similar property extends to the case of polynomials with complex coefficients.

The determination of Fourier phase given modulus has been approached from many viewpoints. Some of the most successful and flexible methods are iterative in nature [149]. One of the most important steps in this area was taken by Gerchberg and Saxton [151]. We use a similar approach. Specifically, the Method of Generalized Projections (MGP) provides a means of dealing with multiple constraints and non-convex sets. One of the principal questions in these cases is that of convergence $[149]$. A property of “error” reduction has been shown for the case of two sets (not necessarily convex) $[149]$.

In our problem both sides of the transform are fundamentally complex (the signal we seek is actually the phase on both sides of the transform). While some work has been done in this area most investigators have been concerned with cases where restriction to a real signal is appropriate. Additionally, our transform is Fresnel and not Fourier. Rolleston et al. [152] consider phase retrieval for real signals from Fresnel transform magnitude. Misell [153] considers recovery of signal phase for complex objects in optics and electron microscopy from two intensity distributions. Fienup et al. [154, 155] consider the reconstruction of complex objects from their Fourier transform magnitude and sufficiently strong signal support constraints. In the area of phase retrieval for optical tomography, Maleki and Devaney use a non-iterative method to compute the phase and reconstruct refractive index profiles of complex weakly scattering objects such as refractive index fibers from two very closely spaced measurements $[156, 157]$. They also discuss Misell-type phase retrieval algorithms for the purpose of tomography in the case of a priori object support information. However, no rigorous
proofs of uniqueness are given [158].

In this thesis the optical intensity is measured over two planes whose complex optical amplitudes are related by a Fresnel transform. In practice, weak scattering assumptions on the acoustic disturbance allow use of a single measurement. The other intensity distribution may be assumed a constant with total energy equal to that in the measured image. The Fresnel transform of an object function may be written essentially as the Fourier transform of the product of the object function and a two-dimensional chirp distribution. A simple modification to the proof given by Hayes [1] is presented in Sec. 3.3.1 extending uniqueness considerations to fully complex signals. We discuss application of this theorem to Fresnel transforms and show that the shift-invariant nature of the Fresnel model limits the size of the signal equivalence class significantly. This is particularly important in effecting a phase recovery in a reasonable number of iterations. It is also important in tomographic applications because the alignment of the signals from various views is critical. The practicality of this technique in recovering the phase for this class of signals is demonstrated via simulation and several experimental data sets. Tomographic reconstruction is performed on several data sets and comparison of a reconstructed complicated double slit pattern to hydrophone measurements is made. Forward and back propagation of views from several different fields is demonstrated in addition to computation of the instantaneous Poynting vector and particle velocity fields.
Measurements are interpreted according to a model for the physical process which produced them. The method for optical measurement of ultrasound fields described in this thesis is essentially the inversion of an optical scattering problem which proceeds from a series of optical intensity measurements. Section 2.1 in this chapter describes the basic physical experiment used to collect the optical intensity data from which the three-dimensional instantaneous pressure distribution is ultimately computed. The measurement process requires calculation of the phase of the scattered optical field from known experimental parameters and the measured optical intensity under a scalar model. Section 2.2 explains the assumptions made in arriving at the scalar optical field model and their relationship to the measurement experiment. Assumptions involving components of the optical imaging system used in the experiment, the structure of the ultrasound field being measured, and wave propagation are made explicit. In this thesis a tomographic reconstruction algorithm is applied to computed optical phase data to reconstruct an acoustically induced instantaneous refractive index perturbation. The algorithm chosen assumes sufficiently weak optical scattering. The implications of this requirement for restrictions on the ultrasound field are discussed in Sec. 2.4. Measurement of the optical intensity is made via a Charge Coupled Device (CCD) array. The discrete nature of this measurement device and the tomographic inversion process imply restrictions on the spatial frequencies present in the optical intensity pattern. An approximate relationship between the temporal frequencies present in the ultrasound waveform and the spatial frequencies present in the optical amplitude and phase functions is derived in Sec. 2.5.

Relating the measured three-dimensional instantaneous refractive index to the space-time functions representing a propagating wide bandwidth acoustic pulse is the subject of sections 2.6 through 2.10. Section 2.3 explains a model for the relationship between instantaneous local refractive index and instantaneous local density. In order to understand the relationship between density, pressure, and particle velocity fields the equation of motion for acoustic propagation are derived from first principles in Sec. 2.6. This equation is nonlinear (even for the case of a linear constitutive relation for the supporting medium) and small signal assumptions for its linearization are discussed. The linear form of the acoustic equations of motion and a linear constitutive relationship from thermodynamics are used to derive the familiar scalar wave equation. The conditions on the angular plane wave spectrum of the ultrasound pulse allowing propagation of the measured instantaneous pressure distribution forward and backward in space and time under the linear model are presented in Sec. 2.7. A sampling theorem is derived in Sec. 2.7.2 that explains the measurements necessary to reconstruct the complete space-time distribution of the sound field without any assumptions on the angular plane wave spectrum. Sections 2.9 and 2.10 describe the computation of particle velocity and energy flow from pressure under the assumptions of Sec. 2.7.

Chapter 7 compares an optical measurement with that obtained via hydrophone. A full three-
Experiment Description

Figure 2.1: A description of the basic experiment. The ultrasound transducer and laser are fired with timing that allows the collision of optical and acoustic pulses to take place near the optical axis of the imaging system. The lenses $\ell_2$ and $\ell_3$ provide a means of imaging arbitrary planes within the imaging volume onto the CCD array. Plane $p_1$ is the plane immediately subsequent to passage of the optical pulse through the acoustic field and is sometimes referred to as the “transducer” plane. Plane $p_2$ is an arbitrary diffraction plane. The dimensions of the water tank used in this experiment are approximately 18 cm × 18 cm × 32 cm.

dimensional hydrophone measurement might be obtained by physically scanning the hydrophone throughout the desired region of space. At each location the ultrasound transducer is fired and the pressure value at that location is recorded for a fixed temporal delay. However, it is also possible to obtain measurements of the full pressure time series over a plane and then propagate this time series to other planes as necessary to obtain a full three-dimensional measurement of the field at a specified time point in the temporal evolution of the sound field. With this technique physical scan time (which is typically the bottleneck) can be spent obtaining high spatial resolution in the plane over which the time series is measured. Section 2.8 describes the relationship between the time series over the hydrophone measurement plane and that over any plane parallel to it.

2.1 Experiment Description

Figure 2.1 shows the basic experimental setup. The principal control signal in the experiment is the vertical synchronization pulse in the National Television Standards Committee (NTSC) video output of the CCD camera. Each vertical synchronization pulse in the NTSC signal indicates the beginning of the exposure period on the alternate field of the CCD array (the camera is interlaced). For each field exposure, the synchronization pulse generator is triggered. The synchronization pulse generator controls the number of times, and the rate at which the acousto-optic scattering experiment is performed during each CCD field integration (exposure). It does so by producing an $N$-cycle square wave with a typical pulse repetition rate of 2 kHz. Each cycle of the square wave triggers a waveform generator which produces the ultrasound waveform and a delayed trigger for the laser firing circuit. The delay on the trigger for the laser firing circuit controls how far the ultrasound pulse propagates before scattering the optical pulse. Each field on the CCD array typically integrates four identical experiments ($N = 4$) at a rate of $f = 2$ kHz. An Image Acquisition (IMAC) board is used to digitize a number of frames for averaging and storage on disk. Typically, fifty frames are averaged. If data for tomographic reconstruction are taken, a stepper motor is used to rotate the ultrasound transducer through 180 degrees about an axis normal to the optical axis of the imaging system.

1The board is a National Instruments PCI-1408 Image Acquisition (IMAC) board. It digitizes with a word length of 8 bits.
2.2 Optical Field Model

Maxwell’s equations provide the basis for understanding the scattering problem inverted to optically measure the acoustic field. In the following sections we briefly review the approximations made in arriving at the model we invert and their relationship to the problem described in this thesis. The discussion is extended from Balanis [159], Goodman [160], and Kong [161].

2.2.1 Maxwell’s Equations

Maxwell’s equations may be stated as follows [159]:

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \mathbf{M} \] (2.1)
\[ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \] (2.2)
\[ \nabla \cdot \mathbf{B} = q_m \] (2.3)
\[ \nabla \cdot \mathbf{D} = q_e \] (2.4)

where

- \( \mathbf{E} \) = Electric field strength (volts/meter)
- \( \mathbf{H} \) = Magnetic field strength (amperes/meter)
- \( \mathbf{B} \) = Magnetic flux density (webers/meter\(^2\))
- \( \mathbf{D} \) = Electric flux density (coulombs/meter\(^2\))
- \( \mathbf{M} \) = Magnetic current density (volts/meter\(^2\))
- \( \mathbf{J} \) = Electric current density (amperes/meter\(^2\))
- \( q_m \) = Magnetic charge density (webers/meter\(^3\))
- \( q_e \) = Electric charge density (coulombs/meter\(^3\))

Taking the divergence of Faraday’s law (Eq. (2.1)), using the fact that the divergence of the curl is always zero together with Gauss’s law (Eq. (2.4)) implies the following conservation of charge relation [161]

\[ \nabla \cdot \mathbf{J} = -\frac{\partial q_e}{\partial t}. \] (2.5)

There are three scalar variables for each of \( \mathbf{E}, \mathbf{H}, \mathbf{B}, \) and \( \mathbf{D} \) comprising a total of twelve unknowns. Equations (2.1)–(2.4) are not independent equations. If we wish to determine all twelve unknowns in \( \mathbf{E}, \mathbf{H}, \mathbf{B}, \) and \( \mathbf{D} \) then twelve independent equations in those variables are needed. Equations (2.3) and (2.4) may be derived from Eqs. (2.1), (2.2), and two conservation laws (Eq. (2.5) and its counterpart for magnetic monopoles). Thus, an additional six scalar equations are needed. For these relationships we turn to the constitutive relations. It is through the constitutive relations that the physical material in which the electromagnetic field exists comes into the model. For this reason there are different sets of constitutive relations for fundamentally different types of materials [161]. In this work only isotropic material needs to be modeled and hence we use the constitutive relations

\[ \mathbf{D} = \epsilon \mathbf{E} \] (2.6)
\[ \mathbf{B} = \mu \mathbf{H}. \] (2.7)

2.2.2 The Wave Equation

We now develop the form of the wave equation from Maxwell’s equations. We first assume no magnetic monopoles, giving \( \mathbf{M} = 0 \) and \( q_m = 0 \). Taking the curl of both sides of Faraday’s law in Eq. (2.1) and using the constitutive relation (2.7) we arrive at

\[ \nabla \times \nabla \times \mathbf{E} = -\left( \nabla \times \frac{\partial}{\partial t} \right) \mu \mathbf{H}. \]
Assuming permeability $\mu$ is constant in space and time and interchanging the order of spatial and temporal differentiation we have

$$\nabla \times \nabla \times \mathcal{E} = -\mu \frac{\partial}{\partial t} \nabla \times \mathcal{H}. \tag{2.8}$$

Using the vector identity (see Marsden and Tromba [162], page 231)

$$\nabla \nabla \cdot A - \nabla^2 A = \nabla \times \nabla \times A$$

on the left hand side of Eq. (2.8), and Ampere’s law (Eq. (2.2)) on the right, brings us to

$$\nabla^2 \mathcal{E} = \mu \frac{\partial^2 D}{\partial t^2} + \mu \frac{\partial \mathcal{J}}{\partial t} + \nabla \nabla \cdot \mathcal{E}.$$

Assuming no impressed current sources and zero conductivity implies $\mathcal{J} = 0$, giving

$$\nabla^2 \mathcal{E} = \mu \frac{\partial^2 D}{\partial t^2} + \nabla \nabla \cdot \mathcal{E}. \tag{2.9}$$

We now rewrite the term $\nabla \nabla \cdot \mathcal{E}$. Assume no electric charge ($q_e = 0$) in the solution region and use Gauss’s law (Eq. (2.4)) and the constitutive relation (2.6) together with the identity for scalar function $f_s$ and vector function $A$,

$$\nabla \cdot f_s A = f_s \nabla \cdot A + A \cdot \nabla f_s \tag{2.10}$$

to obtain

$$\nabla \cdot \mathcal{D} = \epsilon \nabla \cdot \mathcal{E} + \mathcal{E} \cdot \nabla \epsilon = 0$$

and thus,

$$\nabla \cdot \mathcal{E} = -\mathcal{E} \cdot \nabla \ln \epsilon, \tag{2.11}$$

where we have used the scalar identity

$$\nabla \ln f_s = \frac{1}{f_s} \nabla f_s.$$

Using Eq. (2.11) and the constitutive relation (2.6) in Eq. (2.9) gives

$$\nabla^2 \mathcal{E} = \mu \left( \frac{\partial^2 \mathcal{D}}{\partial t^2} \right) \epsilon \mathcal{E} - \nabla \mathcal{E} \cdot \nabla \ln \epsilon. \tag{2.12}$$

Equation (2.12) is a linear, second order partial differential equation for electromagnetic fields in an isotropic, homogeneous medium.

### 2.2.3 Scalar Approximation

Equation (2.12) is a vector equation. It is advantageous to examine why it must be a vector equation and under what approximations or assumptions it might be replaced with a scalar equation. In a spatially homogeneous medium the equation reads

$$\nabla^2 \mathcal{E} = \mu \left( \frac{\partial^2}{\partial t^2} \right) \epsilon \mathcal{E}, \tag{2.13}$$

where $\epsilon$ may be a function of time but not of space. Using the fact that

$$\nabla^2 \mathcal{E} = \nabla^2 (\hat{x} \mathcal{E}_x + \hat{y} \mathcal{E}_y + \hat{z} \mathcal{E}_z) = \hat{x} \nabla^2 \mathcal{E}_x + \hat{y} \nabla^2 \mathcal{E}_y + \hat{z} \nabla^2 \mathcal{E}_z, \tag{2.14}$$
Eq. (2.13) may be written as three separate \textit{scalar} equations of the form
\[
\nabla^2 \mathcal{E}_n = \mu \left( \frac{\partial^2}{\partial t^2} \right) \mathcal{E}_n \quad \text{where } n = x, y, z. \tag{2.15}
\]

If we solve the three scalar wave equations in Eq. (2.15), obtaining \(\mathcal{E}_x, \mathcal{E}_y, \text{ and } \mathcal{E}_z\), a solution to the homogeneous vector wave equation given in Eq. (2.13) is obtained.

Next we consider the case of an approximate solution to Maxwell’s equations. We will assume that static fields are zero. Consider a scalar field \(u(r, t)\) satisfying
\[
\nabla^2 \left( \mathcal{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(r, t) = 0 \tag{2.16}
\]
and let
\[
\mathcal{E} = \hat{x} u(r, t). \tag{2.17}
\]

We now wish to see under what conditions Eq. (2.17) is an approximate solution to the full vector description. Computing the curl of \(\mathcal{E}\) gives
\[
\nabla \times \mathcal{E} = \hat{x} \left( \frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} \right) + \hat{y} \left( \frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} \right) + \hat{z} \left( \frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} \right) \tag{2.18}
\]
allowing computation of \(H\) as
\[
H = -\frac{1}{\mu} \left( \hat{y} \int dt \left( \frac{\partial}{\partial z} \right) - \hat{z} \int dt \left( \frac{\partial}{\partial y} \right) \right) u(r, t). \tag{2.19}
\]

Substituting the \(H\)-field in Eq. (2.19) into Ampere’s law gives
\[
\nabla \times H = \hat{x} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) + \hat{y} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) + \hat{z} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \tag{2.20}
\]

implying
\[
\mathcal{E} = -\frac{1}{\mu c} \int dt \int dt \left\{ \hat{x} \left( \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) + \hat{y} \left( \frac{\partial^2}{\partial x \partial y} \right) + \hat{z} \left( \frac{\partial^2}{\partial x \partial z} \right) \right\} u(r, t). \tag{2.21}
\]

Note that we have liberally interchanged the order of differentiation and integration. It is clear from Eq. (2.21) when a scalar approximation might be valid. We began with a solution to the scalar Eq. (2.16) as the \(\hat{x}\)-component of the \(\mathcal{E}\)-field. All other components were zero. In order for Faraday’s and Ampere’s laws to be \textit{consistent}, the \(\mathcal{E}\)-field obtained in Eq. (2.21) must be the original \(\mathcal{E}\)-field in Eq. (2.16). The second term in the \(\hat{x}\)-component in Eq. (2.21) will return the original \(\mathcal{E}\)-field under the assumption that Eq. (2.16) is satisfied and that second partials in the \(\hat{x}\)– and \(\hat{y}\)-directions may be neglected. Conditions are therefore required under which all other terms may be neglected. Consider a narrow temporal bandwidth field. Integrating with respect to time effectively divides each component by...
A quantity proportional to angular frequency. Thus, energy is coupled into the other terms through partial derivatives. If the variations in the \( \hat{z} \) direction are much higher than those in the other two coordinate directions then the scalar approximation will hold. This occurs when the primary direction of propagation of a very high frequency field is the \( \hat{z} \)-direction, the aperture is sufficiently large so as to not cause significant spreading (spreading of the field would put the spatial variations due to the high temporal frequency of the field in directions other than \( \hat{z} \)), and the field is measured close to the central axis of propagation (paraxial approximation).

Next we look at Gauss’s laws for electric and magnetic charge.

\[
\nabla \cdot \epsilon \mathbf{E} = \epsilon \frac{\partial}{\partial x} u(r, t) \approx 0 \quad (2.22) \\
\nabla \cdot \mu \mathbf{H} \approx \mu \frac{\partial}{\partial y} \int dt \left( \frac{\partial}{\partial z} u(r, t) \right) \approx 0 \quad (2.23)
\]

where we have assumed that \( \mu \) and \( \epsilon \) do not vary appreciably with space. These approximations also depend on the high frequency character of the field and the nature of spatial derivatives and temporal integrals.

2.2.4 Modal Coupling via Inhomogeneities

Energy from one field component in the vector wave Eq. (2.12) is coupled into others via the term involving the natural logarithm. This can be seen from the fact that each of the three scalar equations implied by Eq. (2.12) is a function of all three vector components. If the term involving the natural logarithm is sufficiently small it may be neglected. Without this term the three scalar equations each involve only a single vector component of the total field. Let us look at requirements on the acoustic disturbance we are imaging and the piezooptic coefficient that allow use of such an approximation. Using \( n(r, t) = \sqrt{\epsilon(r, t)} \) we may write the vector wave equation as

\[
\nabla^2 \mathbf{E} = \mu \left( \frac{\partial^2}{\partial r^2} \right) \epsilon \mathbf{E} - 2 \nabla \epsilon \cdot \nabla \ln n(r, t). \quad (2.24)
\]

Assuming a base refractive index of approximately unity and a linear small signal relationship between pressure and density (see Sec. 2.3) the refractive index index variation may be written as

\[
n(r, t) = 1 + p(r, t) \left( \frac{\partial n}{\partial p} \right)_s. \quad (2.25)
\]

Using the Taylor series expansion of the natural logarithm

\[
\ln(1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \ldots
\]

we note

\[
\ln n(r, t) = \ln \left( 1 + p(r, t) \left( \frac{\partial n}{\partial p} \right)_s \right) = p(r, t) \left( \frac{\partial n}{\partial p} \right)_s - \ldots \quad (2.26)
\]

Thus, we require spatial pressure variations and a piezooptic coefficient sufficiently small so as to allow the gradient of the term in Eq. (2.26) to be approximated as zero.

2.2.5 Fresnel Model for Scalar Diffraction

From the above it is clear that under certain restrictions it is appropriate to use a scalar model for electromagnetic wave propagation and scattering. Here we review conditions under which an additional simplification, known as the Fresnel approximation, may be used [1, 2]. Consider the following free space wave equation.

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(r, t) = 0 \quad (2.27)
\]
If \( u(\mathbf{r}, t) \) is time harmonic we assume \( u(\mathbf{r}, t) = \text{Re}\{u(\mathbf{r})e^{j\omega t}\} \) and solve

\[
(\nabla^2 + k^2) u(\mathbf{r}) = 0. \tag{2.28}
\]

Substituting

\[
u(\mathbf{r}) = \int_{-\infty}^{\infty} d\mathbf{r} \ e^{j2\pi \mathbf{k}_\perp \cdot \mathbf{r}} U(\mathbf{k}_\perp, z)
\]

into Eq. (2.28) together with an assumption on the possible directions of propagation for the various plane wave components of the solution yields (see [160])

\[
U(\mathbf{r}_\perp, z) = \int_{-\infty}^{\infty} d\mathbf{k}_\perp \ U(\mathbf{k}_\perp, 0)e^{j\mathbf{k}_\perp \cdot \mathbf{r}}, \tag{2.29}
\]

where \( k_z = \sqrt{k^2 - k^2_\perp}, \) \( k \) is the wave number, and \( \mathbf{k}_\perp = (k_x, k_y) \) is the transverse wave number. Assumption of a band limited spectrum (see [163]) implies \( k^2_\perp \ll k^2 \) and we may write, using the first two nonzero terms in Taylor’s expansion [162]

\[k^2 \sqrt{1 - (k_\perp/k)^2} \approx k - \frac{1}{2} \frac{k^2}{k},\]

giving

\[U(\mathbf{r}_\perp, z) = \int_{-\infty}^{\infty} d\mathbf{k}_\perp \ U(\mathbf{k}_\perp, 0)e^{j(k - \frac{1}{2} \frac{k^2}{k})z}. \tag{2.30}\]

This approximation will be acceptable if the resulting phase error is much less than \( 2\pi \). We may approximate the error \( \alpha_e \) with the third nonzero term in Taylor’s expansion to give

\[
\alpha_e = \frac{1}{8} k^4 z_b/k^3. \tag{2.31}
\]

Given a phase error \( \alpha_e \), Eq. (2.32) defines a maximum distance \( z \) for which the model is valid.

It has been shown (see for example Goodman [160]) that for distances from the aperture sufficiently greater than the wavelength, proper choice of Green’s function yields the following expression of the Huygens-Fresnel principle.

\[
u(\mathbf{r}_\perp, z) = \frac{z}{j\lambda} \int_{-\infty}^{\infty} d\mathbf{r}' u(\mathbf{r}', 0) \frac{e^{jk|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|^2} \tag{2.33}
\]

Here,

\[|\mathbf{r} - \mathbf{r}'| = \sqrt{z^2 + (x - x')^2 + (y - y')^2},\]

describes the distance from the source plane \((z = 0)\) to the point \( \mathbf{r}' \) where the field is to be evaluated. If we are sufficiently distant from the source we may approximate

\[|\mathbf{r} - \mathbf{r}'|^2 \approx z\]

in the denominator of Eq. (2.33). The phase term requires a better approximation, so we again use the first two nonzero terms of Taylor’s series expansion to obtain

\[
\sqrt{z^2 + (x - x')^2 + (y - y')^2} = z \left( 1 + \frac{1}{2} \left( \frac{x - x'}{z} \right)^2 + \frac{1}{2} \left( \frac{y - y'}{z} \right)^2 \right) \approx z \left( 1 + \frac{1}{2} \left( \frac{x - x'}{z} \right)^2 + \frac{1}{2} \left( \frac{y - y'}{z} \right)^2 \right) \tag{2.34}
\]

\[
\approx z \left( 1 + \frac{1}{2} \left( \frac{x - x'}{z} \right)^2 + \frac{1}{2} \left( \frac{y - y'}{z} \right)^2 \right) \tag{2.35}
\]
Refractive Index Model

giving

\[ u(r_{\perp}, z) = \frac{e^{jkz}}{j\lambda z} \int_{-\infty}^{\infty} dr'_{\perp} u(r'_{\perp}, 0) e^{j\frac{k}{\lambda z}((x-x')^2+(y-y')^2)}. \]  

(2.36)

Again we have made an approximation in the phase term and use the third nonzero term in Taylor’s series expansion as an estimate for the error. We write the error \( \beta_c \) as

\[ \beta_c = \frac{1}{8} \frac{a^4 k}{z^3}, \]  

(2.37)

where \( a^2 = x^2 + y^2 \) is the radius of a circle in the \( xy \)-plane that contains the entire aperture. In this case we require the field to be \textit{space limited}.

Using the Fourier transform pair [160, 163]

\[ \frac{e^{jkz}}{j\lambda z} e^{jk(x^2+y^2)/(2z)} \xrightarrow{F} e^{j(k-(1/2)(k^2/k))z} \]

we observe that Eqs. (2.31) and (2.36) are spatial and Fourier domain descriptions of the same model. However, they were derived under assumptions of band-limitedness and space-limitedness respectively. While no signal may be both strictly band and space-limited the required approximations are satisfied in both cases if \( z_s < z_b \). Setting \( \alpha_c = \beta_c \) we obtain the required condition,

\[ a < 8\alpha_c k^2 / k_{\perp}^3, \]  

(2.38)

which allows use of the Fresnel approximation from distances very close to the aperture out to infinity. It should be noted that \( a \) is the aperture of our system, \( k \) is the optical wave number, and \( k_{\perp} \) is the spectral breadth induced into the optical angular plane wave spectrum by the ultrasound field (see Sec. 2.4). Assuming an acoustic wave speed of 1480 m/s, that the highest frequency the ultrasound transducer can pass is 5 MHz, an optical wavelength of \( \lambda = 810 \) nm, and \( \alpha_c = 0.1 \) we obtain a maximum aperture of

\[ a_{\text{max}} = 8\alpha_c k^2 / k_{\perp}^3 = 8 \times 0.1 \times \frac{(2\pi/810 \times 10^{-9})^2}{(2\pi \times 5 \times 10^6/1480)^3} \approx 5 \text{ m}. \]

2.3 Refractive Index Model

Section 2.3.1 describes a simple model for electromagnetic refractive index in a material. A macroscopic approach is taken relating refractive index to the number of molecular dipoles per unit volume in a region. In Sec. 2.3.2 the piezooptic coecient is introduced (see Eq. 2.41) relating local instantaneous refractive index to local instantaneous pressure.

2.3.1 The Acousto-Optic Effect

In a dielectric material the dominant charges in atoms and molecules are bound. That is, they are not free to move about in the material as they are in conductors. Simple descriptions of how such materials interact with electromagnetic fields model the constituent molecules of a medium as charge distributions that distort in the presence of an electric field [159]. This distortion constitutes the formation of an electric dipole. As the electric field oscillates this dipole will follow, emitting a field of its own. The total field in the medium is then the sum of the applied and induced dipole fields. At a macroscopic level this effect may be accounted for by the introduction of a polarization vector \( \mathbf{P} \) [159]. We expect its magnitude to be proportional to the molecular dipole strength and the \textit{number} of molecules per unit volume. Its complex part accounts for absorption, its real part, for the effect of the material on wave speed in the medium. Its units are those of electric flux density and in fact we may write

\[ \mathbf{D} - \mathbf{P} = \epsilon_0 \mathbf{E} \]  

(2.39)
or

\[ D = \epsilon_0 E + P = \epsilon_0 E + \epsilon_0 \chi E = \epsilon_0 (1 + \chi) E, \]

where \( \epsilon_0 \approx 8.85 \times 10^{-12} \text{farads/meter} \) is the permittivity of free space and \( \chi \) is the electric susceptibility. Hence, this model predicts changes in the permittivity of the medium corresponding to changes in the number of molecules per unit volume. The number of molecules per unit volume changes if the density of the material is perturbed by an acoustic wave in the medium. We note that refractive index

\[ n = \sqrt{\epsilon_r} = \sqrt{1 + \chi} \]

is a function of permittivity and thus dipole moment per unit volume under this model. Refractive index is therefore a function of density in a dielectric material. Greater density means more molecules and thus more dipoles per unit volume. A greater number of dipoles per unit volume implies an increase in permittivity and thus refractive index. We expect to observe slower wave speed in areas of greater density. Although greater pressure in a region implies greater density, the two are not generally linearly related. However, under a small signal model their relationship may be considered linear. To first order, refractive index is observed to increase linearly with pressure. It may thus be modeled with a simple coefficient.

2.3.2 The Piezooptic Coefficient

For small pressure variations we may write

\[ n(r, t) = n_o + p(r, t) \left( \frac{\partial n}{\partial p} \right)_s, \]

where the subscript \( s \) indicates that the pressure variations are adiabatic. There are equations which attempt to predict from a theoretical basis the value of the piezooptic coefficient [164,165]. We use the value

\[ \left( \frac{\partial n}{\partial p} \right)_s = 15.1 \times 10^{-12} \text{cm}^2/\text{dyn} \]

measured at 5461 Angstroms by Riley and Klein [164]. This may be converted to more convenient units [166] for our purposes

\[ \left( \frac{\partial n}{\partial p} \right)_s = (15.1 \times 10^{-12} \text{cm}^2/\text{dyn}) \left( 10^5 \text{dyn/N} \right) \left( 10^{-4} \text{m}^2/\text{cm}^2 \right) \]

\[ = 15.1 \times 10^{-12} \text{m}^2/\text{N} \]

\[ = 15.1 \times 10^{-11} \text{Pa}^{-1} \]

Also,

\[ \left( \frac{\partial n}{\partial p} \right)_s = (15.1 \times 10^{-12} \text{cm}^2/\text{dyn}) \left( \frac{1.013 \times 10^6 \text{dyn/cm}^2}{1 \text{ atm}} \right) \]

\[ = 1.52963 \times 10^{-5} \text{ atm}^{-1}. \]

2.4 Optical Scattering Model

In this section the weak scattering model for the interaction of the optical and acoustic pulses is explained. The Fourier Slice Reconstruction (FSR) algorithm (see Sec. 3.4) is used to reconstruct the instantaneous pressure in the ultrasound pulse from optical phase delay data. Thus it is assumed that the ultrasound pulse may be considered a phase object. A phase object induces essentially only phase variations (delays) in the optical wavefront. To consider the sound field a phase object the amount of
Figure 2.2: Planar optical wavefronts impinging on a refractive index object $n_o + \delta n(r)$. The background refractive index is $n_o\). Planes $p_1$ and $p_2$ are parallel to the incident optical wavefronts. For sufficiently small $\delta n(r)$ the optical wave experiences essentially only phase delay (the complex magnitude remains approximately uniform after passage through the sound field represented by $\delta n(r)$). Paths $l_1$ and $l_2$ represent two possible path choices for the line integral in Eq. (2.42). Path $l_1$ represents a path that passes through the refractive index object. Path $l_2$ is any path not passing through any part of the object. The relative phase delay at Plane $p_2$ between any two such paths is desired.

optical scattering must be small. Consider a plane wave impinging on a region of space with varying refractive index $n(r)$ as shown in Figure 2.2. The refractive index variations will induce some breadth into the angular plane wave spectrum, scattering the optical wave. It is desired to compute (assuming a ray optics model) the relative phase delay between points on Plane $p_2$ from a knowledge of $n(r)/c$ (the local value for how many seconds it takes to travel a meter). That is, we desire the relative phase delay between a point on the optical wavefront at Plane $p_2$ lying on a path through the object (see path $l_1$ in Figure 2.2) and a point lying on a path passing only through the background refractive index (see path $l_2$ in Figure 2.2). It is useful to write the refractive index function as

$$n(r) = n_o + \delta n(r)$$

where $\delta n(r)$ is a small change in the local refractive index representing the scatterer. Only differences from ambient or background refractive index need be considered ($n_o$ averages to the same value for all rays). The relative time delay is then

$$t_d = \int \delta n(r)/c \, dl.$$  \hspace{1cm} (2.42)

The refractive index perturbation $\delta n(r)$ goes to zero outside the object. Thus, the integration limits may be restricted to the object support.

Phase delay is related to time delay by frequency. If the total time delay between two points is $t_d$ seconds, then the total phase delay is $t_d \nu$ cycles, where $\nu$ is the frequency of the wave in Hertz. In radians the delay is $t_d \omega$, where $2\pi \nu = \omega$. We may then write the relative phase delay from Eq. (2.42) as

$$\theta = k \int \delta n(r) \, dl,$$

where the wave number $k$ is given by $k = \omega/c$.

\hspace{1cm} 2We might also expect differences in the refractive index as a function of frequency as this would affect the ability of the polarization in the material to follow the electric field.
2.5 Temporal and Spatial Bandwidth Relationships

This section relates the temporal and spatial bandwidths of the field produced by the ultrasound transducer to the spatial frequency content of the forward-scattered optical wave under the scattering model explained in Sec. 2.4. Accurate sampling of the optical intensity in the experiment requires that we have at least an approximate relationship between the possible spatial frequencies present in the refractive index perturbation and the spatial frequency spectrum of the optical wave over Plane \( p_2 \) in Figure 2.2. The frequencies present in the angular plane wave spectrum do not change via propagation in homogeneous material (only the phase relationships between the frequencies change). Thus, we may compute the breadth of the angular plane wave spectrum over Plane \( p_1 \) in Figure 2.1. As this relationship is only approximate the effects of evanescent acoustic field components are not considered.

Let \( B_t \) denote the temporal bandwidth of the transducer. The spatial variations it can produce are related to \( B_t \) by the wave speed via \( \lambda \nu = c \), where \( \nu \) is the temporal frequency of the wave. The spatial frequency of a simple harmonic wave pattern is the reciprocal of the wavelength. Thus

\[
f = \frac{1}{\lambda} = \frac{\nu}{c},
\]

where \( f \) is spatial frequency, \( \lambda \) spatial wavelength, \( c \) wave speed, and \( \nu \) temporal frequency. The lowest spatial frequency is therefore determined by the lowest frequency in \( B_t \) and the highest spatial frequency by the highest frequency in \( B_t \). In this imaging technique, however, the spatial patterns in the acoustic disturbance are not visualized directly but are encoded in the phase of the interrogating optical wave. The relationship between the spatial bandwidth of the acoustic field and the spatial bandwidth of the light field is therefore similar to the relationship between a signal and its phase modulation encoding. Let \( \phi(x, y) \) denote the sufficiently small amplitude spatial pattern of the acoustic field at some instant in time. The optical wave entering this acoustic disturbance is \( e^{j\phi(x, y)} \). We shall neglect Doppler shifting of the EM-wave. The phase modulated optical wave exiting the acoustic field is

\[
A_l = e^{j(kz)} e^{j\phi(x, y)}.
\]

The spatial frequency spectrum is obtained as the Fourier transform of Eq. (2.44). We require that \( \phi(x, y) \) be small under the weak scattering model. Under this condition the relationship between the spectrum of the light field and the sound field is simple. We write

\[
A_l = e^{j(kz)} e^{j\phi_0} e^{j\phi'(x, y)},
\]

where \( \phi'(x, y) \) represents spatial variations of the sound field and \( \phi_0 \) is an average delay. Under the assumption that \( \phi'(x, y) \) is small, a Taylor series and proper choice of phase reference may be used to obtain

\[
A_l = 1 + j\phi'(x, y).
\]

We obtain an approximate spectrum for \( A_l \) by transforming Eq. (2.45). For sufficiently small amplitude pressure variations, the portion of the optical angular plane wave spectrum that represents spatial variations and the corresponding portion of the sound field spectrum are related by the factor \( j \).

We detect the intensity of the field and not the field itself. Hence, the sampled pattern has frequencies that are twice that of the actual field pattern. A minimum spatial sampling rate for the optical intensity image is

\[
s = 4 \max(B_t)/c,
\]

where \( c \) is the wave speed in the medium and \( \max(B_t) \) is the highest temporal frequency passed by the transducer. Evanescent fields have not been considered. Evanescent fields correspond to vibratory patterns with spatial support on the order of a wavelength in size. They are quickly attenuated via propagation. Sound fields considered in this thesis do not have spatial features on the order of an optical wavelength.
2.6 Acoustic Propagation Model

This section leads to the development of a linear wave equation for acoustic pressure (see Eq. (2.61)). The assumptions and approximations in the development starting from the Newtonian principles of conservation of mass and momentum are explained and the concept of pressure is defined precisely (see Eq. (2.58)). The resulting equation is used in the propagation of a pressure time series measured over a plane (see Secs. 2.8 and 7), in computation of particle velocity (see Sec. 2.10) and Poynting vector (see Sec. 2.9) fields, and in the propagation of measured spatial distributions (see Secs. 2.7).

Let us consider the development of the equation of motion for an elastic solid and the concept of power flow. Here elastic solid refers to a material whose internal forces may be accurately modeled by means of a second rank tensor to be discussed later and wherein particles maintain a certain uniformity of relative position (see page 21).

We begin with the law of conservation of mass. This is an important part of the development as it involves a mapping that makes the statement of the law trivial but bears explanation because of its importance in subsequent steps.

Consider a particle of material at some location \( \mathbf{r} \) in a Cartesian coordinate system. This particle undergoes motion (acceleration) due to forces acting on it and becomes displaced or moves to a new location \( \mathbf{q} \). We may write the relationship between \( \mathbf{q} \) and \( \mathbf{r} \) as

\[
\mathbf{q}(t) = \mathbf{r} + \xi(t),
\]

where \( \xi(t) \) is a vector valued function describing the displacement from \( \mathbf{r} \) as a function of time. It is not necessary that \( \xi(t) \) ever be zero. In this case, the particle is never at \( \mathbf{r} \) and this location becomes simply an arbitrary, fixed reference point. It will be useful later, however, if we define \( \xi(0) = 0 \), so that \( \mathbf{r} \) is clearly the location at \( t = 0 \). Two particles can not occupy the same space at the same time. Thus, the stipulation \( \xi(0) = 0 \) makes the location of each particle at time \( t = 0 \) a unique label for that particle. It also means that the displacement function labeled for a particle may be labeled with the location of that particle at \( t = 0 \) as well. Thus we may write

\[
\mathbf{q}(\mathbf{r}, t) = \mathbf{r} + \xi(\mathbf{r}, t). \tag{2.47}
\]

Note that \( \xi(\mathbf{r}, t) \) and \( \mathbf{q}(\mathbf{r}, t) \) are both vector valued functions describing vectors in a Cartesian coordinate system. We require physically that \( \mathbf{q}(\mathbf{r}_1, t) \neq \mathbf{q}(\mathbf{r}_2, t) \) for \( \mathbf{r}_1 \neq \mathbf{r}_2 \) making the mapping \( \mathbf{q}(\mathbf{r}, t) \) one-to-one (see [162], page 367). For every \( \mathbf{q}(\mathbf{r}, t) \) there is clearly a corresponding location at \( t = 0 \), \( \mathbf{q}(\mathbf{r}, 0) = \mathbf{r} \), making the mapping onto as well (see [162], page 369). We may assume continuity of \( \xi(\mathbf{r}, t) \) given that the material is not too severely taxed. Equation (2.47) thus describes a "well-behaved", invertible mapping.

Now consider what happens to a well-behaved (integrable) volume of points under the mapping of Eq. (2.47). Clearly, since the mapping is one-to-one and onto the volume evolves in time in an invertible fashion. It will be necessary later in the development for the surface of this volume to have a well defined normal everywhere. We therefore require that there exist a parameterization of the surface such that the surface is considered smooth.

The next question is that of the continuity of the mapping. We use the definition of Marsden and Tromba [162], page 107.

**Definition 1 (Continuity)** Let \( f : A \subset \mathbb{R}^n \to \mathbb{R}^m \) be a given function with domain \( A \). Let \( \mathbf{x}_0 \in A \). We say \( f \) is continuous at \( \mathbf{x}_0 \) iff

\[
\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0).
\]

If we say that \( f \) is continuous we shall mean that \( f \) is continuous at each point \( \mathbf{x}_0 \) of \( A \).

Hence, if the mapping is continuous, proximity at time \( t = 0 \) implies proximity at time \( t \). Points remain, in some sense, near their neighbors. A point inside the surface at \( t = 0 \) will remain "inside" the surface as it evolves in time. The requirement of continuity limits the kind of particle motion that can be considered. Some physical phenomena will not be modeled in this fashion. For example,
Acoustic Propagation Model

some kinds of flow and mixing problems do not satisfy this continuity requirement. We call this an elastic solid model, as moderately taxed solids do not have particles changing positions and acquiring new neighbors.

We are now ready to state the law of conservation of mass. Consider a volume at time $t = 0$. Denote this volume by $\Omega(t)$ and define $\Omega_0 = \Omega(0)$. Note that $\Omega(t)$ is obtained by simply applying the map $q(r, t)$ to each point in $\Omega_0$. Since “inside” remains “inside” for all time we have

$$\left(\frac{d}{dt}\right) \int_{\Omega(t)} d\mathbf{r} \rho(\mathbf{r}, t) = 0,$$

(2.48)

where $\rho(\mathbf{r}, t)$ is density as a function of spatial coordinate $\mathbf{r}$ at time $t$. Equation (2.48) simply states that the total mass inside $\Omega(t)$ is a constant. That constant is simply the mass that was there at time $t = 0$, hence

$$\int_{\Omega(t)} d\mathbf{q} \rho(\mathbf{q}, t) = \int_{\Omega_0} d\mathbf{r} \rho(\mathbf{r}, 0).$$

(2.49)

We intend to develop an identity involving the integrands in Eq. (2.49). The change of variables theorem may be used to make the region of integration for each integral the same. For the left hand side of Eq. (2.49) we have

$$\int_{\Omega(t)} d\mathbf{q} \rho(\mathbf{q}, t) = \int_{\Omega_0} d\mathbf{r} J(q(\mathbf{r}, t), t) \rho(q(\mathbf{r}, t), t),$$

(2.50)

where the Jacobian

$$J(q(\mathbf{r}, t), t) = \frac{\partial q(\mathbf{r}, t)}{\partial \mathbf{r}} = I + \begin{bmatrix} \frac{\partial}{\partial x} \xi_1(\mathbf{r}, t) & \frac{\partial}{\partial y} \xi_2(\mathbf{r}, t) & \frac{\partial}{\partial z} \xi_3(\mathbf{r}, t) \\ \frac{\partial}{\partial y} \xi_1(\mathbf{r}, t) & \frac{\partial}{\partial z} \xi_2(\mathbf{r}, t) & \frac{\partial}{\partial x} \xi_3(\mathbf{r}, t) \\ \frac{\partial}{\partial z} \xi_1(\mathbf{r}, t) & \frac{\partial}{\partial x} \xi_2(\mathbf{r}, t) & \frac{\partial}{\partial y} \xi_3(\mathbf{r}, t) \end{bmatrix},$$

and $I$ is the $3 \times 3$ identity matrix. This gives

$$\int_{\Omega_0} d\mathbf{r} J(q(\mathbf{r}, t), t) \rho(q(\mathbf{r}, t), t) - \rho(\mathbf{r}, 0) = 0.$$

(2.51)

The volume of integration is arbitrary, allowing us to write

$$J(q(\mathbf{r}, t), t) \rho(q(\mathbf{r}, t), t) = \rho(\mathbf{r}, 0).$$

(2.52)

Next we look at conservation of momentum and the stress tensor. We define the stress tensor via the diagram shown in Figure 2.3. We use the mean value theorem and assume the element to be small allowing us to consider the stress at the center of the element. The applied stress at the center of the element is a description of the force per unit area on each projection in each of the three coordinate directions yielding a nine component tensor. Internal or balancing stresses are the negative of applied stress. From now on when we refer to stress we are speaking of internal stress. Let $T_{ij}$ denote the force per unit area in the $j$-direction on the projection whose normal is parallel to the $i$-axis. The unit normal on the original surface element describes the area of each of these three projections succinctly. Let $A_i$ be the area of the projection with normal parallel to the $i$-axis, $dS$ the area of the original surface element, and $n_i$ the $i$th component of the unit normal. Then $A_i = n_i dS$ and the total force on the surface patch is

$$F_j = -\sum_i A_i T_{ij} = -\sum_i n_i T_{ij} dS$$

or

$$\mathbf{F} = -\mathbf{T} \cdot \mathbf{n} dS,$$
The total force on the surface bounding $\Omega(t)$ is therefore

$$F = -\int_{\partial\Omega(t)} \mathbf{T} \cdot d\mathbf{S} = -\int_{\partial\Omega(t)} d\mathbf{S} \cdot \mathbf{T}.$$ 

We may now apply Newton's law $F = (d/dt)\mathbf{m}$, where $\mathbf{m}$ is momentum, to arrive at

$$\left(\frac{d}{dt}\right) \int_{\Omega(t)} d\mathbf{q} \rho(\mathbf{q}, t) \left(\frac{\partial}{\partial t}\right) \mathbf{q}^{-1}(\mathbf{q}, t) = -\int_{\partial\Omega(t)} d\mathbf{q} \mathbf{T}(\mathbf{q}, t) \cdot \nabla q,$$

where $\nabla q = [\partial/\partial u, \partial/\partial v, \partial/\partial w]^T$. We want to move the time derivative inside the integral and so apply the change of variables theorem to both sides of Eq. (2.53). In preparation for this, an expression for the divergence operator in terms of $\mathbf{r}$ (as opposed to $\mathbf{q}$) is needed. The chain rule states

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u} \frac{\partial}{\partial x} + \frac{\partial y}{\partial u} \frac{\partial}{\partial y} + \frac{\partial z}{\partial u} \frac{\partial}{\partial z}.$$ 

In our case

$$\mathbf{q}(\mathbf{r}, t) = \mathbf{r} + \mathbf{q}(\mathbf{r}, t) = \{u(\mathbf{r}, t), v(\mathbf{r}, t), w(\mathbf{r}, t)\},$$

where $\mathbf{r} = \{x, y, z\}$ and $u(\mathbf{r}) = x + \xi_u(\mathbf{r}, t)$, etc. We write

$$
\begin{bmatrix}
  \left(\frac{\partial}{\partial u}\right) \\
  \left(\frac{\partial}{\partial v}\right) \\
  \left(\frac{\partial}{\partial w}\right)
\end{bmatrix}
= 
\begin{bmatrix}
  \frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} + \frac{\partial z}{\partial u} \\
  \frac{\partial x}{\partial v} + \frac{\partial y}{\partial v} + \frac{\partial z}{\partial v} \\
  \frac{\partial x}{\partial w} + \frac{\partial y}{\partial w} + \frac{\partial z}{\partial w}
\end{bmatrix}
= 
\begin{bmatrix}
  \left(\frac{\partial x}{\partial u}\right) & \left(\frac{\partial y}{\partial u}\right) & \left(\frac{\partial z}{\partial u}\right) \\
  \left(\frac{\partial x}{\partial v}\right) & \left(\frac{\partial y}{\partial v}\right) & \left(\frac{\partial z}{\partial v}\right) \\
  \left(\frac{\partial x}{\partial w}\right) & \left(\frac{\partial y}{\partial w}\right) & \left(\frac{\partial z}{\partial w}\right)
\end{bmatrix}.
$$
Acoustic Propagation Model

or $\nabla_q = \mathbf{A} \nabla_r$. Applying the change of variables theorem to Eq. (2.53) yields

$$
\left( \frac{d}{dt} \right) \int_{\Omega_o} d\mathbf{r} J(\mathbf{q}(\mathbf{r}, t), t) \rho(\mathbf{q}(\mathbf{r}, t), t) \frac{\partial}{\partial t} \mathbf{q}(\mathbf{r}, t) = -\int_{\Omega_o} d\mathbf{r} \mathbf{T}(\mathbf{q}(\mathbf{r}, t), t) \cdot (\mathbf{A} \nabla_r) J(\mathbf{q}(\mathbf{r}, t), t)
$$

(2.54)

Using Eq. (2.52) and moving the time derivative inside the integral gives

$$
\int_{\Omega_o} d\mathbf{r} \rho(\mathbf{r}, 0) \frac{\partial^2}{\partial t^2} \mathbf{q}(\mathbf{r}, t) + \mathbf{T}(\mathbf{q}(\mathbf{r}, t), t) \cdot (\mathbf{A} \nabla_r) J(\mathbf{q}(\mathbf{r}, t), t) = 0.
$$

(2.55)

The region of integration in Eq. (2.55) is arbitrary and so we may write

$$
\rho(\mathbf{r}, 0) \frac{\partial^2}{\partial t^2} \mathbf{q}(\mathbf{r}, t) = -\mathbf{T}(\mathbf{q}(\mathbf{r}, t), t) \cdot (\mathbf{A} \nabla_r) J(\mathbf{q}(\mathbf{r}, t), t)
$$

(2.56)

which is a nonlinear equation.

Under the assumption that displacement changes slowly within the volume (its derivatives are small) $\mathbf{A} \approx \mathbf{I}$ and $J(\mathbf{q}(\mathbf{r}, t), t) \approx 1$. In addition, if it is assumed that the stress does not vary too rapidly in space and that displacements are sufficiently small we may write

$$
\rho(\mathbf{r}, 0) \frac{\partial^2}{\partial t^2} \mathbf{q}(\mathbf{r}, t) = -\mathbf{T}(\mathbf{q}(\mathbf{r}, t), t) \cdot (\nabla_r)
$$

(2.57)

which is linear.

Next we simplify the stress tensor to model an ideal liquid (no support of shear stress) by setting all off diagonal terms to zero and placing the average of the diagonal elements along its diagonal, giving

$$
\mathbf{T}(\mathbf{r}, t) = p(\mathbf{r}, t) \mathbf{I},
$$

(2.58)

where $p(\mathbf{r}, t) = [T_{xx}(\mathbf{r}, t) + T_{yy}(\mathbf{r}, t) + T_{zz}(\mathbf{r}, t)]/3$. This defines the concept of pressure. From Eq. (2.58) we write

$$
\mathbf{T}(\mathbf{r}, t) \cdot \nabla_r = \nabla p(\mathbf{r}, t).
$$

We may thus write Eq. (2.57) as a relationship between pressure and velocity yielding,

$$
\rho(0, \mathbf{r}) \frac{\partial}{\partial t} \mathbf{v}(\mathbf{r}, t) = \nabla p(\mathbf{r}, t).
$$

(2.59)

Equation (2.59) is actually three equations in four unknowns. In order to complete the description of the material motion we need, as in electromagnetics, a constitutive relation describing the relationship between displacement (or velocity) and stress (or pressure) for a particular material. We take from thermodynamics a relationship between the divergence of velocity, the temporal derivative of pressure, and the compressibility $\kappa(\mathbf{r})$

$$
\kappa(\mathbf{r}) \frac{\partial}{\partial t} p(\mathbf{r}, t) = -\nabla \cdot \mathbf{v}(\mathbf{r}, t).
$$

(2.60)

We may now develop a wave equation for pressure as follows. Take the divergence of Eq. (2.59) to obtain

$$
\rho_v \frac{\partial}{\partial t} \nabla \cdot \mathbf{v}(\mathbf{r}, t) = -\nabla \cdot \nabla p(\mathbf{r}, t),
$$

where we have assumed a homogeneous density. Now substitute from Eq. (2.60) to arrive at

$$
\rho_v \kappa \frac{\partial^2}{\partial t^2} p(\mathbf{r}, t) = \nabla^2 p(\mathbf{r}, t)
$$

(2.61)

which we recognize as a wave equation with $c = 1/\sqrt{\rho_v \kappa}$. Equation 2.61 forms the basis for all subsequent discussions of acoustic wave propagation in this thesis.


2.7 Propagation of Spatial Distribution

Given a measurement of the three-dimensional instantaneous pressure in a wide bandwidth acoustic disturbance, what is known about the pressure distribution at other times? Section 2.7.1 shows that under the free-space wave equation a forward-propagating assumption on the wide bandwidth angular plane wave spectrum of the disturbance allows us to compute the entire four-dimensional space-time dependence of the pressure function. In this context, the term “forward-propagating” means that there exists a plane such that all wave vector components in the direction of the plane normal are greater than zero (equivalently, they may all be less than zero). In Sec. 2.7.2 a new sampling theorem is derived stating that two complete measurements of the three-dimensional pressure field separated in time by $t$ allow release of the forward-propagating assumption for every acoustic wave number $k$ satisfying $k \neq n\pi/(c\Delta t)$, where $c$ is the acoustic wave speed in the medium and $n$ an integer greater than zero. Thus, the temporal evolution of very general ultrasound fields may be computed from a minimal number of measurements.

2.7.1 Single Measurement in Time

Consider the free-space wave equation

$$\Box^2 u(r, t) = 0 \quad (2.62)$$

where

$$\Box^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}.$$ 

We wish to determine $u(r, t)$ from an initial measurement of $u(r, 0)$ and a simple condition on allowable wave numbers. Expand $u(r, t)$ in a complex exponential basis

$$u(r, t) = \int_{-\infty}^{\infty} dk \, e^{ik \cdot r} U(k, t).$$

Substitute into Eq. (2.62) and use the linearity of the integral and wave operators (together with the fact that the wave operator affects only the temporal and spatial coordinate variables) to arrive at

$$\Box^2 u(r, t) = \Box^2 \int_{-\infty}^{\infty} dk \, e^{ik \cdot r} U(k, t)$$

$$= \int_{-\infty}^{\infty} dk \, \Box^2 e^{ik \cdot r} U(k, t)$$

$$= -\int_{-\infty}^{\infty} dk \, e^{ik \cdot r} \left( k^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) U(k, t)$$

$$= 0 \quad (2.63)$$

where $k$ represents the magnitude of $k$. The complex exponentials are a complete basis and as such Eq. (2.63) implies

$$\left( k^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) U(k, t) = 0.$$ 

We may choose a typical basis for the solution space of this problem such as

$$U(k, t) = W_1(k) e^{-ikct} + W_2(k) e^{ikct}$$

and form the spatial position domain representation

$$u(r, t) = \int_{-\infty}^{\infty} dk \, W_1(k) e^{-ik \cdot r} e^{ikct} + \int_{-\infty}^{\infty} dk \, W_2(k) e^{ik \cdot r} e^{ikct} \quad (2.64)$$
Each of the integral terms in Eq. (2.64) may be interpreted as a weighted sum of plane waves. Each plane wave has a direction in space determined by the wave vector \(k\). Let us orient the transducer parallel to the \(xy\)-plane. Note that each term in Eq. (2.64) has component plane waves which move both directions across the plane of the transducer emitting surface. That is, both terms have waves of the form

\[ p(r, t) = e^{\pm j(k \cdot r + c t)} \quad k_z > 0. \]

Auxiliary conditions for the determination of the weighting functions \(W_{1,2}(k)\) are more conveniently applied to a more physically intuitive basis such as

\[ U(k, t) = W_f(k)e^{-j \text{sign}(k_z)kct} + W_b(k)e^{j \text{sign}(k_z)kct} \]

where

\[ \text{sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \]

Thus the spatial position domain representation becomes

\[ u(r, t) = \int_{-\infty}^{\infty} dk W_f(k)e^{-j \text{sign}(k_z)kct}e^{jk \cdot r} + \int_{-\infty}^{\infty} dk W_b(k)e^{j \text{sign}(k_z)kct}e^{jk \cdot r}. \] (2.65)

Equation (2.65) has two terms, each consisting of plane waves directed in only one direction across the plane containing the emitting surface of the transducer. We assume that the pulse has no plane wave components with \(k_z \leq 0\) and thus \(W_b(k) = 0\) giving

\[ u(r, t) = \int_{-\infty}^{\infty} dk W_f(k)e^{-j \text{sign}(k_z)kct}e^{jk \cdot r}. \] (2.66)

Next we note that at \(t = 0\)

\[ u(r, 0) = \int_{-\infty}^{\infty} dk W_f(k)e^{jk \cdot r}, \] (2.67)

or

\[ W_f(k) = \int_{-\infty}^{\infty} dk u(r, 0)e^{-jk \cdot r}. \] (2.68)

Hence,

\[ u(r, t) = \int_{-\infty}^{\infty} dk U(k, 0)e^{-j \text{sign}(k_z)kct}e^{jk \cdot r}. \] (2.69)

There are no evanescent waves in the source-free wideband solution given in Eq. (2.69). It should be noted that their presence in the solution of the time-harmonic (Helmholtz) wave equation is a result of not constraining the boundary or initial values to be solutions of the source-free equation and that they represent the presence of a source. It should also be noted that the ability to assign spatial variations in a field to these decaying source terms is a direct result of the assumptions about the temporal spectrum of the field. If it is assumed monochromatic we may easily assign spatial variations not satisfying \(\omega^2/c^2 = k^2\), where \(k\) is the Fourier transformation variable, to an evanescent (exponential decay) portion of the model. If the field has a broad temporal spectrum (a broad range of \(\omega\)) observation of a full time series (in a plane) would be needed in order to properly assign spatial and temporal variations to propagating and evanescent portions of a model.
2.7.2 Measurement at Multiple Distinct Time Points

We now prove a theorem explaining the measurements necessary to reconstruct the complete spacet ime distribution of the sound field without any assumptions on the angular plane wave spectrum. From two measurements of the spatial distribution at times $t_1$ and $t_2$ we may write (using the basis in Eq. (2.64))

$$U(k, t_1) = W_1(k)e^{-jkc t_1} + W_2(k)e^{jkc t_1}$$
$$U(k, t_2) = W_1(k)e^{-jkc t_2} + W_2(k)e^{jkc t_2}.$$  \hspace{1cm} (2.70)

Writing Eq. (2.70) as a matrix equation gives

$$\begin{bmatrix} U(k, t_1) \\ U(k, t_2) \end{bmatrix} = \begin{bmatrix} e^{-jkc t_1} & e^{jkc t_1} \\ e^{-jkc t_2} & e^{jkc t_2} \end{bmatrix} \begin{bmatrix} W_1(k) \\ W_2(k) \end{bmatrix}$$

or

$$u = Mw.$$  \hspace{1cm} (2.71)

Equation (2.7.2) has a unique solution if the matrix $M$ is non-singular. We compute the determinant

$$|M| = e^{jkc(t_2 - t_1)} - e^{-jkc(t_2 - t_1)} = 2i \sin(kc(t_2 - t_1)),$$

and note that $M$ is non-singular when

$$t_2 - t_1 \neq n \frac{T}{2},$$

where $T = \pi/(kc)$ is the temporal period corresponding to a wave number of $k$. That is, for each wave number $k$ we may solve for $w(k)$ unless the time difference in the measurements is some integer multiple of half the temporal period corresponding to $k$.

2.8 Propagation of Time Series

In this section we derive the relationship between a pressure time series measured over a plane and the time series over any other plane under a linear, scalar model for wave propagation. This relationship allows the propagation of a measured time series to other planes and is thus very useful in obtaining experimental data with minimal measurements.

Consider the free space wave equation

$$\Box^2 u(r, t) = 0$$  \hspace{1cm} (2.72)

where

$$\Box^2 \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}.$$  

We wish to determine $u(r, t)$ from an initial measurement of the time series over some plane $z = z_0$ and a simple condition on allowable wave numbers. Expand $u(r, t)$ in a complex exponential basis

$$u(r, t) = \int_{-\infty}^{\infty} d\omega \int d\omega_1 e^{j\omega_1 \cdot r} e^{-j\omega t} U(k_\perp, \omega, z).$$

Substitute Eq. (2.8) into Eq. (2.72) and use the linearity of the integral and wave operators (together with the fact that the wave operator affects only the temporal and spatial coordinate variables) to
arrive at

\[ \square^2 u(r, t) = \square^2 \int_{-\infty}^{\infty} d k_\perp d \omega e^{jk_\perp \cdot r_\perp} e^{-j\omega t} U(k_\perp, \omega, z) \]

\[ = \int_{-\infty}^{\infty} d k_\perp d \omega \square^2 e^{jk_\perp \cdot r_\perp} e^{-j\omega t} U(k_\perp, \omega, z) \]

\[ = -\int_{-\infty}^{\infty} d k_\perp d \omega e^{jk_\perp \cdot r_\perp} e^{-j\omega t} \left( k^2 - k_\perp^2 + \frac{\partial^2}{\partial z^2} \right) U(k_\perp, \omega, z) \]

\[ = 0, \quad (2.73) \]

where we have used the Fourier representation of the free-space wave equation \( kc = \omega \). Equation (2.73) implies that

\[ \left( k^2 - k_\perp^2 + \frac{\partial^2}{\partial z^2} \right) U(k_\perp, \omega, z) = 0. \]

Hence we may write,

\[ U(k_\perp, \omega, z) = W_f(k_\perp, \omega)e^{jk_z z} + W_b(k_\perp, \omega)e^{-jk_z z}, \]

where \( k_z = \sqrt{k^2 - k_\perp^2} \). Thus

\[ u(r, t) = \int_{-\infty}^{\infty} d k_\perp d \omega W_f(k_\perp, \omega)e^{jk_z z} e^{jk_\perp \cdot r_\perp} e^{-j\omega t} + \int_{-\infty}^{\infty} d k_\perp d \omega W_b(k_\perp, \omega)e^{-jk_z z} e^{jk_\perp \cdot r_\perp} e^{-j\omega t}. \]

(2.74)

Under the assumption that the field may be represented as a superposition of waves of the form

\[ p(r, t) = e^{jk_z z} e^{jk_\perp \cdot r_\perp} e^{-j\omega t} \]

we set \( W_b(k_\perp, \omega) = 0 \) and note

\[ u(r_\perp, t, 0) = \int_{-\infty}^{\infty} d k_\perp d \omega W_f(k_\perp, \omega)e^{jk_\perp \cdot r_\perp} e^{-j\omega t} \]

or

\[ W_f(k_\perp, \omega) = \int_{-\infty}^{\infty} d k_\perp d \omega u(r_\perp, t, 0)e^{-jk_\perp \cdot r_\perp} e^{j\omega t}. \]

Thus

\[ u(r, t) = \int_{-\infty}^{\infty} d k_\perp d \omega U(k_\perp, \omega, 0)e^{jk_z z} e^{jk_\perp \cdot r_\perp} e^{-j\omega t}. \]

(2.75)

We observe the presence of evanescent fields in this model (due to the term \( k_z = \sqrt{k^2 - k_\perp^2} \)). This is in contrast to the source-free model in Sec. 2.7.

### 2.9 Power Flow

An important concept in the understanding of acoustic fields is that of power flow. The Poynting vector field describes the direction and magnitude of the local instantaneous rate of energy flow per unit area. This section explains how it relates to the local particle velocity and stress or pressure fields. Equation 2.76 below is used in Chapter 8 to compute Poynting vector fields from experimental data.

**Optical Measurement of Wide Bandwidth Ultrasound Fields**
The amount of work per unit area per particle is found as the integral of the dot product of force per unit area with distance. We write

\[ W_p(\mathbf{r}, t) = \int_{\sigma(t)} \mathbf{T}(\mathbf{q}(\mathbf{r}, t), t) \cdot d\mathbf{r}, \]

where \( \sigma(t) \) is the path of the particle at \( \mathbf{r} \) when \( t = 0 \) and \( d\mathbf{r} \) is a vector of differential length tangent to the particle path. From the definition of a line integral (see [162], page 421) we have

\[ W_p(\mathbf{r}, t) = \int_0^t \mathbf{T}(\mathbf{q}(\mathbf{r}, \tau), \tau) \cdot \frac{\partial}{\partial \tau} \mathbf{\xi}(\mathbf{r}, \tau) d\tau = \int_0^t \mathbf{T}(\mathbf{q}(\mathbf{r}, \tau), \tau) \cdot \mathbf{v}(\mathbf{r}, \tau) d\tau \]

The instantaneous power flow per particle is then

\[ \mathbf{T}(\mathbf{r}, t) = \frac{\partial}{\partial t} W_p(\mathbf{r}, t) = \mathbf{T}(\mathbf{q}(\mathbf{r}, t), t) \cdot \mathbf{v}(\mathbf{r}, t) \]

Under the assumptions that result in a linear equation of motion we may write

\[ \mathcal{I}(\mathbf{r}, t) = \mathbf{T}(\mathbf{r}, t) \cdot \mathbf{v}(\mathbf{r}, t) \]

In an ideal fluid, using the concept of pressure we have

\[ \mathcal{I}(\mathbf{r}, t) = p(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t). \]  

### 2.10 Calculation of Particle Velocity From Pressure Field

Local instantaneous particle velocity is a quantity of interest in the measurement and study of acoustic fields. In a homogeneous, isotropic medium the velocity and pressure fields are related via an impedance concept. Knowledge of the full space-time distribution of one allows computation of the other. If the full space-time dependence is not available, in some cases it may be inferred from what is known (see Secs. 2.7.1, 2.7.2, and 2.8). This section develops a relation that allows the computation of local instantaneous particle velocity from local instantaneous pressure under the assumption that the pressure field possesses a forward-propagating angular plane wave spectrum. The term forward-propagating means wave vector components normal to the emitting surface of the transducer are all greater than zero. This assumption is discussed in detail in Sec. 2.7.1. An assumption on the source-free nature of the data is also needed (there are no evanescent fields in the wide band model). The relationship in Eq. (2.84) is used in Chapter 8 to compute local instantaneous particle velocity from experimental data. Section 2.7.2 describes a sampling theorem allowing the elimination of the forward-propagating assumption for multiple pressure field measurements.

We begin with propagation of an initial measurement of three-dimensional instantaneous pressure (see Eq. (2.69) in Sec. 2.7.1).

\[ p(\mathbf{r}, t) = \int_{-\infty}^{\infty} dk \ P(k, 0) e^{-j \text{sign}(k_z) k c t} e^{j k \cdot \mathbf{r}} \]  

We intend to use (see Eq. (2.59))

\[ \rho_0 \frac{\partial}{\partial t} \mathbf{v}(\mathbf{r}, t) = -\nabla p(\mathbf{r}, t) \]  

\[ (2.78) \]

to compute \( \mathbf{v}(\mathbf{r}, t) \). From Eqs. (2.77) and (2.78) we write

\[ \frac{1}{\rho_0} \nabla p(\mathbf{r}, t) = \frac{1}{\rho_0} \nabla \int_{-\infty}^{\infty} dk P(k, 0) e^{-j \text{sign}(k_z) k c t} e^{j k \cdot \mathbf{r}} \]

\[ = \frac{j}{\rho_0} \int_{-\infty}^{\infty} dk k P(k, 0) e^{-j \text{sign}(k_z) k c t} e^{j k \cdot \mathbf{r}} \]

\[ = -\frac{\partial}{\partial t} \mathbf{v}(\mathbf{r}, t). \]  

\[ (2.79) \]
Also,

\[ \int_0^t \frac{\partial}{\partial \tau} \mathbf{v}(\mathbf{r}, t) d\tau = \mathbf{v}(\mathbf{r}, t) - \mathbf{v}(\mathbf{r}, 0). \]

The last term is simply a constant velocity component which we will assume to be zero. This gives

\[
\mathbf{v}(\mathbf{r}, t) = -\frac{j}{\rho_o} \int_0^t \int_{-\infty}^{\infty} d\mathbf{k} \mathbf{k} P(\mathbf{k}, 0) e^{-j \text{sign}(k_z) k x t} e^{j \mathbf{k} \cdot \mathbf{r}} \quad (2.82)
\]

\[
= \frac{1}{\rho_o} \int_{-\infty}^{\infty} d\mathbf{k} \frac{\text{sign}(k_z)}{k c} k P(\mathbf{k}, 0) e^{-j \text{sign}(k_z) k x t} e^{j \mathbf{k} \cdot \mathbf{r}}. \quad (2.83)
\]

Using \( c = 1/\sqrt{\kappa \rho_o} \), we have

\[
\mathbf{v}(\mathbf{r}, t) = \sqrt{\frac{\kappa}{\rho_o}} \int_{-\infty}^{\infty} d\mathbf{k} \text{sign}(k_z) \hat{k} P(\mathbf{k}, 0) e^{-j \text{sign}(k_z) k c t} e^{j \mathbf{k} \cdot \mathbf{r}}, \quad (2.84)
\]

where \( \hat{k} = \mathbf{k}/||\mathbf{k}|| \).
Several signal processing concepts and algorithms germane to the optical method for measurement of wide bandwidth ultrasound fields discussed in this thesis are explained in this chapter. Section 3.1 explains the relationship between the optical intensity of the forward-scattered optical field and the digitized representation of this signal recorded and stored on disk during the experiment. It is important to understand this relationship because it must be used to obtain data that are directly proportional to the actual optical intensity in the physical experiment before proceeding with phase retrieval via the Method of Generalized Projections (MGP) (see Sec. 3.3.2). Section 3.2 describes a two-step method for reducing image background corruption in the optical intensity recording resulting from dust particles in the beam path, defects in the optical elements, nonuniformity of the optical beam, multiple reflections in the optical imaging path, and similar deterministic sources of error that vary slowly as the ultrasound field is rotated during the data collection process.

The ray sum of instantaneous pressure (taken in the propagation direction of the incident optical pulse) is reconstructed from the phase of the optical pulse immediately after passage through the sound field. This two-dimensional optical phase function is computed from the intensity of the forward-scattered optical field measured in a plane located a known distance from the scattering event. It is found as the solution of a fixed point equation (see Eq. (3.46)) via the Method of Generalized Projections (MGP) as described in Sec. 3.3.2. The ambiguity in this reconstructed phase is minimal. Only a very small class of signals satisfies the fixed point equation making the computed phase function unique up to a constant, unit magnitude, complex coefficient (see Sec. 3.3.1). Section 3.4 discusses the Fourier Slice Reconstruction (FSR) algorithm used to reconstruct the instantaneous pressure distribution from several two-dimensional optical phase functions. This set of phase functions is obtained by performing the basic measurement experiment at several angles as the ultrasound field is rotated about an axis normal to the propagation direction of the incident optical pulse. The spatial sampling rate for the optical intensity images and the angular sampling rate necessary for accurate reconstruction of pressure in the ultrasound pulse are computed in this section as well. The nullspace of the sampling operator and its relevance to the finite support oscillatory patterns common in ultrasound fields is discussed (see Figure 3.10).

### 3.1 Video Signal Digitization

To recover the shape of the optical wavefront it is very important that data proportional to the actual optical intensity in the diffraction plane (Plane $p_2$ in Figure 2.1) are obtained. Otherwise, the phase for a different physics experiment is recovered. The most important stages of the data recording process are shown in Figure 3.1.
Background Correction

Figure 3.1: A description of the data path from integration on the CCD array to analog to digital conversion by the image acquisition board.

Charge Coupled Device (CCD) arrays are very linear within their range of operation. After conversion to National Television Standards Committee (NTSC) video signal format and transmission via coaxial cable to the Image Acquisition (IMAC) board the signal is digitized using the affine mapping,

\[ s = \left( \frac{v_{\text{in}} - v_{\text{min}}}{v_{\text{max}} - v_{\text{min}}} \right) \cdot \frac{255}{255} + v_{\text{min}} \]

where \( v_{\text{max}} \) and \( v_{\text{min}} \) are maximum and minimum reference voltages respectively. These may be set on a per experiment basis. It is necessary to correctly invert this mapping via

\[ v_{\text{in}} = s \left( \frac{v_{\text{max}} - v_{\text{min}}}{255} \right) + v_{\text{min}} \quad \text{(3.1)} \]

to obtain data having a proper relationship to the actual physical experiment.

In addition to the affine mapping performed by the IMAC board, the CCD darkfield image must be compensated for. The darkfield image is the signal sent by the CCD camera in the absence of any incident photons. Ideally, this would be zero. The darkfield image is denoted by \( d(m, n) \). The maximum and minimum reference voltages used to record the darkfield image and the signal image are typically not the same. Hence, the darkfield image is compensated for after inverting the affine mapping performed by the IMAC board via

\[ v_{\text{sig}}(m, n) = v_{\text{in}}(m, n) - d(m, n) \quad \text{(3.2)} \]

where \( v_{\text{in}}(m, n) \) and \( d(m, n) \) have been mapped to a signal proportional to the NTSC output via Eq. (3.1).

### 3.2 Background Correction

#### 3.2.1 Basic

In a physical optical system there are several sources of signal corruption. Dust particles on the lenses and optical flats produce Airy patterns. Aperture effects create non-ideal conditions near the edge of the beam (as well as limiting resolution). Multiple reflections create multipath artifacts. For example, they can introduce bright disks or points into the image as the lens system brings backscattered portions of the imaging beam partially into focus in the imaging plane. Such artifacts are particularly troublesome in tomographic imaging as they lead to view-to-view similarities between the errors. These errors then reconstruct as concentric rings centered about the tomographic axis of rotation. This ring artifact can be very pronounced, severely degrading the quality of the measurement.

Phase aberration introduced into the imaging beam in one plane yields, after propagation, amplitude corruption in another plane and vice versa. In the absence of the sound field, imaging a particular plane onto the CCD array and recording the intensity yields an image representing all of the phase aberration, multipath corruption, etc. mapped to the image plane. Consider Plane \( p_1 \) in Figure 3.2. Imaging this plane onto the CCD array yields an image containing background corruption introduced by all the optical elements, not merely the collimating lens \( \ell_1 \) and optical flat \( f_1 \). Let us assume that the structure of the multipath corruption is not changed significantly by the introduction of the sound field (an assumption consistent with the weak scattering model). In this case the corruption observed
Figure 3.2: A depiction of the optical imaging path. The laser source (not shown) is in the left focal plane of collimating lens $\ell_1$. After collimation the principle path of the optical wave moves through the optical flat $f_1$ on the left of the water tank, the ultrasound field, Plane $p_1$, Plane $p_2$, the optical flat $f_2$ on the right side of the water tank, and the imaging lenses $\ell_2$ and $\ell_3$ before impinging on the CCD array at the far right. Dust on the lenses, optical flats, and CCD array, non-ideal aspects of the laser source, and multiple reflections between the several optical elements in the imaging path lead to deterministic signal corruption in the recorded intensity image.

when imaging the optical intensity in Plane $p_1$ may be considered multiplicative. Equation (3.2.1) shows the relationship between measured optical intensity $I(r, z_t)$ in Plane $p_1$ (whose location along the optical axis of the imaging system in Figure 3.2 is denoted by $z_t$), the multiplicative corruption with amplitude $a_c(r, z_t)$ and phase $\theta_c$, and the desired signal with amplitude $a_s(r, z_t)$ and phase $\theta_s$.

\[
I(r, z_t) = |u_s(r, z_t)u_c(r, z_t)|^2 = |a_s(r, z_t)a_c(r, z_t)\exp(i(\theta_s + \theta_c))|^2 = |a_s(r, z_t)a_c(r, z_t)|^2 = I_s(r, z_t)I_c(r, z_t) \tag{3.3-3.6}
\]

The background intensity corruption may then be removed with

\[
I_s(r, z_t) = I_{ms}(r, z_t)/I_{mns}(r, z_t), \tag{3.8}
\]

where $I_{ms}(r, z_t)$ is the intensity measured in Plane $p_1$ (the “transducer plane”) with the ultrasound present and $I_{mns}(r, z_t)$ the intensity measured in the absence of the ultrasound pulse. We have used $I_{mns}(r, z_t) \approx I_c(r, z_t) = I_s(r, z_t) \approx 1$ from the weak scattering assumption.

In a diffraction plane (Plane $p_2$ in Figure 3.2) the relationship between the background and data images is different. We observe the effects of propagation on a disturbance that was introduced in another plane. However, again, the change in the multipath is assumed small. That is, corruption in the diffraction plane is assumed unaffected by the presence of the sound field induced delay (in the optical wavefront). Again, a simple multiplicative model is used. If no magnification is needed in the system it is possible to construct an imaging experiment with no optics subsequent to the interaction of the optical wave with the ultrasound pulse thus greatly reducing the difficulties with background corruption. 

Another possible method of dealing with the signal corruption is to take both background and data images in Plane $p_1$ and Plane $p_2$ in Figure 3.2. The phase retrieval algorithm may then be applied to

\[\text{Reducing corruption from optical elements is one of the principal motivations behind Gabor holography.}\]
the background image pair and to the data image pair separately, allowing subsequent computation of the difference in their phases. Clearly, this is an approximation because multiple reflections are not explicitly taken into account in the model. It does not work appreciably better than the above model and requires twice as many phase retrievals.

3.2.2 Ring or Ridge Artifact Suppression

Ring artifacts are a common source of problems in tomography. To understand their possible sources let us consider how a simple point object appears under Radon transformation. A point in the spatial position domain moves circularly about the rotational axis during tomographic data collection producing a sinusoid in the Radon transform domain. The amplitude of this sinusoid specifies the radius of the circle physically traced out by the point (its distance from the axis of rotation) while the phase specifies its angular location in the spatial position domain. Now consider a point-like source of error that is consistent with view angle. Examples of this are an improperly calibrated sensor, optical amplitude variations caused by multiple reflections, stationary Airy patterns in the image resulting from small particles in the beam path, etc. Such corruption will result in a ridge (or ridges) in the Radon transform domain parallel to the angular coordinate axis. It should be noted that a real object produces a Radon transform with properties different from such ridges. For example the zero and 180 degree views of a real object are related by a reflection and a shift. Consistent view to view corruption is the same in the zero and 180 degree images. During the reconstruction process the one-dimensional Fourier transform of each view is computed. Due to the linearity of the Fourier transform, similarity between views is retained. The addition of the same local feature in each view may be considered an identical perturbation of one-dimensional Fourier coefficients. This results in circular rings or ridges in the two-dimensional Fourier transform domain of the object distribution. Enforcing the conjugate symmetry of real objects in the Fourier domain does not eliminate the artifact. The conjugate symmetric part of the artifact remains and reconstructs to ring-like features centered about the tomographic axis of rotation in the spatial position domain. It is important to understand that in our experiment the Radon transform is not measured directly. Rather, intensity is measured as a function of angle and a set of phase functions representing the Radon Transform is computed. It is the similarity of features between the intensity and the retrieved phase that allow us to understand ring artifacts in the above fashion. Figure 3.5 shows the intensity measured along the same row of the CCD array as a function of angle. We note the following salient features of the ridge corruption (which assume a mildly additive model for the ridges).

- If we consider measured optical intensity as a function of CCD array element (in a given row of the CCD array) and angle, the artifacts originate as streaks or ridges parallel to the angular axis.
The primary locations of smaller scale ridges parallel to the angular coordinate in the two-dimensional Fourier plane are shaded. In a typical two-dimensional wavelet filter bank application there are four images generated. Each image corresponds to a set of marked regions in the Fourier plane. The image retaining frequencies in the region marked LL may be viewed as a low frequency approximation to the original image. It is referred to as the approximation image. High frequency details are contained in all of the other regions. The region marked HL contains ridge-like features that run parallel to the angular coordinate. The region marked LL may be filtered again, producing another set of four images. Each subsequent filter level operates on the approximation image from the last.

- The ridges exist at different scales (depending on the extent and structure of the corruption that produced them) but are generally limited to smaller scales.
- The ridges extend through nearly the entire angular coordinate and appear to be a \textit{weak} function of the undulations of the actual data.

The above indicate that a ridge will have Fourier components typically contained in the shaded regions of Figure 3.4.

We consider a filter based on two-dimensional wavelet decomposition of the image for the following reasons.

- We may view wavelet decomposition in terms of two-dimensional filter banks that tend to separate Fourier components of an image in a manner similar to that shown in Figure 3.4.
- Recursive application of two-dimensional wavelet decomposition will allow us to handle ridges at a few of the smaller scales. For this application Symlets of order 5 were determined to work well empirically.

Figure 3.5 shows the result of applying a wavelet filter to intensity as a function of angle and CCD array element number (along a single row of the CCD array).

### 3.3 Signal Phase Retrieval

Phase retrieval belongs to a class of general image recovery or reconstruction problems that deals with the recovery of a function from partial knowledge of the function and its transform (where the transform relationship is known). Typically, the transform is Fourier and our knowledge consists of the magnitude or phase of the Fourier transform and some characteristics on the other side of the transform such as realness or region of support. The recovery-from-phase problem is linear. The recovery-from-magnitude problem is not. The problem discussed in this thesis is somewhat atypical in that the transform is Fresnel, the function is fully complex on both sides of the transform and the
Figure 3.5: A row of the CCD array for several angles of transducer rotation (normal to the optical propagation direction). The upper image is unfiltered. The lower image was filtered by zeroing the first three levels of coefficients in the region marked HL in Figure 3.4.
magnitudes of both members of the transform pair are known (although Misell considered a similar problem [153]). Three questions typically arise in the solution of problems of this type. The first is that of existence. If the transform pair magnitudes had been chosen arbitrarily, the Fresnel model were inadequate, or if the measurements were sufficiently noisy the existence question would play a more central role in our discussion. However, in practice the conditions and measurements are sufficient that the transform relationship remains a good model. The next question is that of uniqueness. Under what conditions does knowledge of signal magnitude, signal transform magnitude, and region of support, uniquely specify a transform pair and how useful are these conditions? Section 3.3.1 introduces the concept of a signal equivalence class and derives conditions under which a signal satisfying constraints on transform magnitude and region of support will be restricted to such a class. The third question is that of solution stability. Given a method of solution, what are the conditions for convergence and what kind of convergence can be expected? Section 3.3.2 explains an iterative solution known as the Method of Generalized Projections (MGP). The problem discussed in this thesis possesses a set distance reduction property (see page 45) guaranteeing a monotonic decrease in a metric known as the Summed Distance Error (SDE) (see page 45) under application of the MGP.

3.3.1 Signal Uniqueness

We may make some use of the extensive work on uniqueness that has been done on the Fourier recovery-from-magnitude problem by viewing the Fresnel transform as a Fourier transform of the product of an object function and a chirp [160]. This relationship is clear from the expression for Fresnel propagation in Eq. (3.9).

$$u(x, y) = \frac{e^{j k z} e^{j(x^2+y^2)k/(2z)}}{j \lambda z} \int_{-\infty}^{\infty} d\xi d\eta \left[ u(\xi, \eta)e^{j(\xi^2+\eta^2)k/(2z)} \right] \exp \left\{ -j \frac{2\pi}{\lambda z} (x\xi + y\eta) \right\} \quad (3.9)$$

Knowledge of the distance $z$ (the distance between Planes $p_1$ and $p_2$ in Figures 2.1 and 3.2) determines the complex scale factor in front of the integral in Eq. (3.9). The function in square brackets on the right of Eq. (3.9) and the function $u(x, y)$ on the left are Fourier transform pairs to within a scale factor. The functions $u(x, y)$ and $u(\xi, \eta)$ are Fresnel transform pairs. The significance of the presence of the chirp function in the Fresnel transform is investigated later (see page 42). We assume adequately sampled continuous functions and so consider the case of a discrete spatial position domain function. We now extend a result by Hayes [167] to include the case of fully complex signals on both sides of the Fourier transform.

We examine the uniqueness of a spatial position domain object function in terms of Fourier transform magnitude via the $z$-transform. We first introduce some basic concepts and definitions. The $m$-dimensional $z$-transform of a sequence is a polynomial in the $m$ variables $z = (z_1, z_2, \ldots, z_m)$ given by

$$X(z) = \sum_{n_1=1}^{N_1} \cdots \sum_{n_m=1}^{N_m} x(n_1, \ldots, n_m)z_1^{-n_1} \cdots z_m^{-n_m}. \quad (3.10)$$

We define a vector with non-negative integer elements $n = (n_1, \ldots, n_m)$ and write

$$z^n = z_1^{n_1}z_2^{n_2} \cdots z_m^{n_m}.$$ 

Equation (3.10) may now be written more compactly as

$$X(z) = \sum_n x(n)z^{-n}.$$ 

All sequences $x(n)$ are assumed to have $z$-transforms with a region of convergence that includes the unit ball $|z_k| = 1$ for $k = 1, 2, \ldots, m$, so that the Fourier transform may be obtained as

$$X(\omega) = X(z) \bigg|_{z = e^{j\omega}} = \sum_n x(n)e^{-jn\omega}.$$
Comparison between vectors of equal length \( N \), \( \mathbf{n} < \mathbf{m} \) is defined to mean that \( n_k < m_k \) for \( k = 1, \ldots, N \). Furthermore, it will be assumed without loss of generality that all sequences \( x(\mathbf{n}) \) are zero for \( n < 0 \). A sequence \( x(\mathbf{n}) \) that is zero outside the region \( 0 \leq n \leq N \) is said to have support \( R(\mathbf{N}) = R(N_1, N_2, \ldots, N_m) \). The field of complex numbers is denoted by \( \mathcal{C} \). The statement \( x \in \mathcal{C}(\mathbf{n}) \), where \( \mathbf{n} \) have \( m \) components, means that \( x(\mathbf{n}) \) is a complex \( m \)-dimensional sequence with causal support \( (x(\mathbf{n}) \text{ is nonzero only for } n \geq 0) \). We will have use for the following theorem from algebra.

**Theorem 1 (Unique Polynomial Factorization)** Any polynomial \( p \in \mathcal{F}(z) \) of nonzero degree may always be factored into a product of polynomials which are irreducible over the field \( \mathcal{F} \). Furthermore, if \( p(z) \) has two different factorizations

\[
p(z) = f_1(z)f_2(z) \cdots f_k(z) = g_1(z)g_2(z) \cdots g_m(z)
\]

then \( m = k \) and the factors \( f_i(z) \) and \( g_i(z) \) may be ordered in such a way that the factors are associated (differ only by factors of zero degree).

Thus \( X(z) \) may be written uniquely (to within factors of zero degree) as the Hadamard product

\[
X(z) = \alpha z^{-\mathbf{n}_0} \prod_{k=1}^{p} X_k(z), \quad (3.11)
\]

where \( \mathbf{n}_0 \) is a vector of nonnegative integers, \( \alpha \) a complex coefficient, and \( X_k(z) \) for \( k = 1, 2, \ldots, p \) are nontrivial irreducible polynomials in \( z^{-1} \). The trivial factors represent shifts and hence contain information about the location of the object in the spatial position domain.

Consider the following \( z \)-transform of a finite support signal \( \{a_0, a_1, a_2, \ldots, a_N\} \).

\[
X(z) = \sum_{n=0}^{N} a_n z^{-n} = a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}
\]

Then

\[
z^{-N} X^*(1/z^*) = z^{-N} \sum_{n=0}^{N} a_n^* z^n = a_0^* z^{-N} + a_1^* z^{-(N-1)} + a_2^* z^{-(N-2)} + \ldots + a_N^*
\]

is the \( z \)-transform of \( \{a_N^*, a_{N-1}^*, a_{N-2}^*, \ldots, a_0^*\} \) or the conjugate time-reversed sequence. This concept may be extended to multidimensional sequences and we define

\[
\tilde{X}(z) = z^{-\mathbf{n}} X^*(1/z^*) \quad (3.12)
\]

as the conjugate space-reversed sequence. Clearly, the function \( \tilde{X}(z) \) will always be a polynomial in \( z^{-1} \).

We define a **conjugate symmetric** \(^2\) \( z \)-transform as follows

**Definition 2 (Conjugate Symmetry)** The \( z \)-transform of a sequence \( x \in \mathcal{C}(\mathbf{n}) \), where \( \mathcal{C}(\mathbf{n}) \) is the set of all, complex, \( n \)-dimensional sequences with finite, causal support, is conjugate symmetric if, for some vector \( \mathbf{k} \) of positive integers

\[
X(z) = \pm z^{-\mathbf{k}} X^*(1/z^*) \quad (3.13)
\]

For example, the sequence \( \{a_0, a_1, a_2, a_2^*, a_1^*, a_0^*\} \) is conjugate symmetric because the same sequence is obtained by reversal and conjugation. A conjugate symmetric sequence has a Fourier transform with phase of the form \( \exp(j\omega n) \). This is referred to as linear phase and thus in the \( z \)-transform domain terms of the form \( z^n \) are sometimes referred to as linear phase terms. The concept of conjugate symmetry expresses itself in the \( z \)-domain via the structure of the zeros (or in higher dimensions zero contours) of the polynomial factors in Eq. (3.13). It is clear from Theorem 1 and Eq. (3.13) that if a

\(^2\)Note that symmetric includes what is sometimes called anti-symmetric.
sequence is conjugate symmetric then its zeros (or zero contours) occur in *conjugate reciprocal pairs*. Finite support, conjugate symmetric sequences have the following property.

\[ \tilde{X}(z) = \pm X(z) \]  

(3.14)

Consider the autocorrelation function

\[ r(n) = x(n) * x^*(-n). \]

Because \(|X(\omega)|\) is the Fourier transform of \(r(n)\) knowing the Fourier transform magnitude is equivalent to knowing the autocorrelation sequence \(r(n)\). Using the convolution theorem we write

\[ R(z) = X(z)X^*(1/z^*). \]  

(3.15)

We may use Eqs. (3.11) and (3.15) to write

\[ R(z) = |\alpha|^2 \prod_{k=1}^{p} X_k(z)X_k^*(1/z^*). \]  

(3.16)

Notice the cancellation of linear phase factors which destroys information about location in the position domain. Following Hayes [167] we form a somewhat more convenient function from \(R(z)\) that contains the same information about \(x(n)\) as

\[ Q(z) = z^{-N}R(z) = |\alpha|^2 \prod_{k=1}^{p} X_k(z)\tilde{X}_k(z). \]  

(3.17)

Let us consider the information contained in Eq. (3.17). Using the convolution theorem together with the definitions of the tilde sequence and conjugate symmetry, we see that the irreducible factors of \(Q(z)\) define a set of sequences in the position domain whose convolution yields (to within a factor of zero degree) the autocorrelation sequence \(r(n)\). For each sequence in this set there is another sequence that is its conjugate space-reversed counterpart. We may look at these sequences or their \(z\)-transform domain representations as the basic building blocks for \(r(n)\). Thus, asking if a sequence possesses a reducible \(z\)-transform is equivalent to asking if there are smaller sequences which may be convolved to yield the original. If we knew which factors (or equivalently which subsequences) to choose we could reconstruct \(x(n)\). The conditions under which phase retrieval is possible are simply those which allow unambiguous separation of the irreducible factors of \(Q(z)\) into those belonging to \(x(n)\) and those corresponding to their tilde counterparts.

Hayes [167] provided a theorem covering the case of a real spatial domain signal. We now offer a simple extension to this theorem for the case of a complex spatial domain signal. To the best of the author’s knowledge it is not found in the literature.

**Theorem 2 (Z-Transform Factorization)** Let \(x \in \mathbb{C}(n)\) and let \(X(z)\) be given by

\[ X(z) = \alpha z^{-n_0} \prod_{k=1}^{p} X_k(z) \]  

(3.18)

where \(X_k(z)\) for \(k = 1, 2, \ldots, p\) are nontrivial irreducible polynomials. If \(y \in \mathbb{C}(n)\) and \(|X(\omega)|\) and \(|Y(\omega)|\) for all \(\omega\) the \(Y(z)\) must be of the form

\[ Y(z) = |\alpha|e^{i\theta}z^{-m} \prod_{k \in I_1} X_k(z) \cdot \prod_{k \in I_2} \tilde{X}_k(z) \]

where \(I_1\) and \(I_2\) are complementary subsets of the integers in the interval \([1, p]\).

The proof is as follows. Assume

\[ |X(\omega)| = |Y(\omega)| \]  

(3.19)
and that the components of \( N \) are large enough so that both \( y(n) \) and \( x(n) \) are zero outside the region of support \( R(N) \). From Eq. (3.19) the autocorrelation functions of the sequences \( x(n) \) and \( y(n) \) are equal and

\[
X(z)X^*(1/z^*) = Y(z)Y^*(1/z^*). \tag{3.20}
\]

Let \( y(n) \) have a z-transform given by

\[
Y(z) = \beta z^{-m_0} \prod_{k=1}^{q} Y_k(z). \tag{3.21}
\]

Equations (3.20), (3.18), and (3.21) imply

\[
|\alpha|^2 \prod_{k=1}^{p} X_k(z)X_k^*(1/z^*) = |\beta|^2 \prod_{k=1}^{q} Y_k(z)Y_k^*(1/z^*). \tag{3.22}
\]

Multiplying both sides of Eq. (3.22) by \( z^{-N} \) gives

\[
|\alpha|^2 z^{-m_1} \prod_{k=1}^{p} X_k(z)X_k^* = |\beta|^2 z^{-m_2} \prod_{k=1}^{q} Y_k(z)Y_k^*. \tag{3.23}
\]

All factors on both sides of Eq. (3.23) are polynomials in \( z^{-1} \). Also, \( m_1 \geq 0 \) and \( m_2 \geq 0 \) (see Eq. (3.12)). Theorem 1 implies \( m_1 = m_2 \), \( p = q \), and that the factors on both sides of Eq. (3.23) may be ordered so that \( Y_k(z) \) is associated with either \( X_k(z) \) or \( \hat{X}_k(z) \). Thus \( Y(z) \) is of the form

\[
Y(z) = \eta z^{-m} \prod_{k \in I_1} X_k(z) \prod_{k \in I_2} \hat{X}_k(z). \tag{3.24}
\]

The linear phase term, \( z^{-m} \), in Eq. (3.24) remains unspecified because the autocorrelation operation destroys information about shifts. We note that \( \eta \) is a generic zero degree complex coefficient because Theorem 1 only requires that the factors on both sides of Eq. (3.23) can be ordered so that they are associated. Associated polynomials may differ by a factor of zero degree. We may use Eq. (3.19) and the fact that space-reversal and conjugation does not change Fourier transform magnitude to observe that \( |\eta| = |\alpha| \). Thus Eq. (3.24) becomes

\[
Y(z) = |\alpha| e^{i\theta} z^{-m} \prod_{k \in I_1} X_k(z) \prod_{k \in I_2} \hat{X}_k(z). \tag{3.25}
\]

We now discuss the implications of Theorem 2 for the recovery of \( x(n) \) from its autocorrelation sequence \( r(n) \) or equivalently its \( Q(z) \) function. In general we do not have enough information to recover \( X(z) \) uniquely. We lack any information about the linear phase terms in Eq. (3.25) and the unit magnitude complex coefficient specified by \( \theta \). These however, do not affect the shape of the recovered image. A more serious problem is that we do not know how to choose between an irreducible factor of \( Q(z) \) and its conjugated space-reversed counterpart. Consider the case of a single irreducible factor (\( p = 1 \)). We have only two choices. We choose the factor of \( Q(z) \) belonging to \( X(z) \) or the conjugate space-reversed counterpart of the factor. However, the basic shape of the sequence remains unchanged under conjugate space-reversal (or conjugate reflection through the origin as it is sometimes called) and so we obtain useful information about \( x(n) \). Thus, in the case of a single, irreducible factor we may obtain \( x(n) \) up to a complex unit magnitude constant (\( \theta \) is unknown), a shift (the linear phase factors are unknown), and a conjugate reflection through the origin. Now consider what happens if we replace a conjugate symmetric factor with its conjugate space-reversed counterpart. Equation (3.14) indicates that at most we would alter \( X(z) \) and hence \( x(n) \) by a sign. This difference may be absorbed into the unit magnitude complex factor determined by the unknown \( \theta \). Hence, if the irreducible factors of \( Q(z) \) are all conjugate symmetric then \( x(n) \) may be recovered up to a complex unit magnitude constant and a shift.
Let us consider the case were \( Q(z) \) (or equivalently \( X(z) \)) possesses at most one non-conjugate symmetric irreducible factor. The product of conjugate symmetric factors is clearly a conjugate symmetric polynomial. Let \( g_1(n) \) be conjugate symmetric and \( g_2(n) \) be non-conjugate symmetric. Let \( g'_2(n) \) be the conjugate space-reversed \( g_2(n) \) so that \( g'_2(n) = g'_2(-n) \). Also, let \( g(n) = g_1(n) * g_2(n) \). We observe

\[
g'(n) = g_1(n) * g'_2(n) = g_1(n) * g_2(-n) = g'_1(-n) * g'_2(-n) = g'(-n)
\]

Thus, if there is at most one non-conjugate symmetric irreducible factor using its conjugate space-reversed counterpart will simply conjugate space-reverse the final convolution product. We may define an equivalence class as follows

**Definition 3 (Equivalence)** We say that \( y(n) \) is equivalent to \( x(n) \) if for some real, scalar \( \theta \) and some vector \( k \) with integer components

\[
y(n) = \begin{cases} 
e^{-\theta} x(k + n) \\ e^{\theta} x^*(k - n) \end{cases}
\]

and write \( y \sim x \)

We may now state

**Theorem 3** Let \( x \in \mathcal{C}(n) \) have a \( z \)-transform with at most one irreducible non-conjugate symmetric factor, i.e.,

\[
X(z) = P(z) \prod_{k=1}^{p} X_k(z)
\]

where \( P(z) \) is irreducible and where \( X_k(z) \) for \( k = 1, \ldots, p \) are irreducible and conjugate symmetric. If \( y \in \mathcal{C}(n) \) with \( |X(\omega)| = |Y(\omega)| \), then \( y \sim x \).

Theorem 3 does not imply that any phase paired with the Fourier transform magnitude of \( x(n) \) produces a position domain signal of the same shape \( (y \sim x) \). Rather, only signals satisfying the constraints of the theorem produce a signal in the same equivalence class. The phase must belong to a finite support sequence.

Let \( \mathcal{C}(n, k) \) be the set of all polynomials of degree \( n \) in \( k \) variables with complex coefficients. It has been shown that the subset of \( \mathcal{C}(n, k) \) consisting of all reducible polynomials in \( \mathcal{C}(n, k) \) corresponds to a set of measure zero in \( R^{2\alpha(n, k)} \) where \( \alpha(n,k) \) is the number of coefficients required to define a polynomial in \( \mathcal{C}(n, k) \) [148]. In other words, “almost all” polynomials in two or more variables are irreducible. Thus the set of reducible polynomials is in some sense sparse. However, it is important to realize that two sparse sets may have very different structures [149]. For example, consider the subset of the reals formed by the rational numbers. This set has measure zero. However, for any element not in the set it is possible to find a member of the set arbitrarily close to it. If the geometric character of the set of reducible polynomials with complex coefficients is similar to that of the rationals then we might expect the irreducibility of a polynomial to be very sensitive to noise and measurement errors.

It has been shown for the case of polynomials with real coefficients that the geometric character for the set of reducible polynomials is not that of the rationals among the reals. Rather, the irreducibility property of a polynomial with real coefficients is such that it provides a stable framework for the retrieval of phase from magnitude [149, 150]. The general success of phase retrieval in this research will be taken to mean that a similar property extends to the case of polynomials with complex coefficients.
We now consider the implications of using a Fresnel transform as in Eq. (3.9) as opposed to Fourier. We may apply Theorem 3 to the product

\[ x(n) = u(n \xi, n \eta) e^{j(n \xi^2 + n \eta^2)k/(2z)}, \]

(3.30)

where \( n \xi \) and \( n \eta \) are integers.\(^3\) If the \( z \)-transform of the product in Eq. (3.30) has at most one irreducible non-conjugate symmetric factor then any finite support sequence \( y(n) \) such that \(|X(\omega)| = |Y(\omega)|\) satisfies

\[ y(n) = \begin{cases} 
    e^\theta u(m \xi + n \xi, m \eta + n \eta) e^{j((m \xi + n \xi)^2 + (m \eta + n \eta)^2)k/(2z)} & \\
    e^{\theta} u^*(m \xi - n \xi, m \eta - n \eta) e^{-j((m \xi - n \xi)^2 + (m \eta - n \eta)^2)k/(2z)}. 
\end{cases} \]

However, the form of the chirp in Eq. (3.30) does not need to be recovered. It takes the form shown in the upper case of Eq. (3.30) and is built into the phase retrieval algorithm through use of the Fresnel transform. Therefore, there is no shift and no conjugate reflection through the origin, giving

\[ y(n) = e^{\theta} u(n \xi, n \eta) e^{j(n \xi^2 + n \eta^2)k/(2z)}. \]

This is a result of the shift invariance property exhibited by the wave equation and the Fresnel approximation. Shift invariance is not exhibited by the Fraunhofer approximation (Fourier transform). Thus, given a finite support, complex signal \( u(n \xi, n \eta) \) for which the \( z \)-transform of \( u(n \xi, n \eta) \exp\{j(n \xi^2 + n \eta^2)k/(2z)\} \) possesses at most a single non-trivial, irreducible factor, knowledge of the Fresnel transform magnitude specifies the signal up to a unit magnitude complex coefficient.

### 3.3.2 Phase Retrieval via Generalized Projections

If the induced refractive index variation is sufficiently small, optical scattering may be modeled as a simple accumulative phase delay (see Sec. 2.4). We refer to the sound field as a phase object denoting the fact that the magnitude of the optical wave remains essentially unchanged. In such a case the scattered optical field immediately after exiting the sound field has phase given by

\[ \theta = k \int \delta n(r) \, dl, \]

(3.31)

where \( \delta n(r) \) is a small change in the local refractive index. The phase of the optical field therefore represents a line integral of the instantaneous pressure in the ultrasound pulse. A complete set of line integrals for 180 degree rotation of the ultrasound field about an axis normal to the direction of optical wave propagation constitutes the Radon transform of the pulse.

The data planes of our experiment are shown in Figure 3.6. The magnitude is recorded in both Planes \( p_1 \) and \( p_2 \). The complex distributions over Planes \( p_1 \) and \( p_2 \) comprise a Fresnel transform pair. If the function satisfies the conditions discussed in Sec. 3.3.1 there is sufficient information to define the phase over Plane \( p_1 \). In this research a type of Gerchberg-Saxton algorithm is used to recover the phase of the optical wave in Plane \( p_1 \). The algorithm begins with the magnitude of the field in Plane \( p_1 \) and an estimate for the phase. This field estimate is then propagated to Plane \( p_2 \) and any known constraints on the field over this plane are applied. In this case, the intensity over Plane \( p_2 \) in Figure 3.6 has been measured, and thus the magnitude is known. The computed phase is retained. The new field estimate is then propagated back to Plane \( p_1 \) where a magnitude constraint is again applied. In our model the acoustic pulse is a phase object and therefore the magnitude in Plane \( p_1 \) is always unity. After verifying this experimentally we may, in practice, only record the data in plane \( p_2 \). The algorithm stops when the corrected (measured) magnitude and the predicted magnitude in one of the planes are not significantly different.

Much research has been done on the method of Projection Onto Convex Sets (POCS). However, in our experiment the sets constrain the magnitude of the functions over Planes \( p_1 \) and \( p_2 \) are not convex. The algorithm therefore falls into the category of restoration by the Method of Generalized

\(^3\)In using this equation to represent physical problems it should be noted that the quotient \( k/(2z) \) is scaled to allow integer valued indexing of the functions.
Figure 3.6: A description of the basic experiment. The ultrasound transducer and laser are fired with timing that allows the collision of optical and acoustic pulses to take place near the optical axis of the imaging system. The lenses \( \ell_2 \) and \( \ell_3 \) provide a means of imaging arbitrary planes within the imaging volume onto the CCD array. Plane \( p_1 \) is the plane immediately subsequent to passage of the optical pulse through the acoustic field and is sometimes referred to as the “transducer” plane. Plane \( p_2 \) is an arbitrary diffraction plane. The dimensions of the water tank used in this experiment are approximately \( 18 \text{ cm} \times 18 \text{ cm} \times 32 \text{ cm} \).

Figure 3.7: (a) A convex set. A line segment may be drawn between any two points in the set with all points on the line lying in the set. (b) A non-convex set. There exist two points in the set for which a line segment connecting them contains points not in the set.

Projections (MGP). In order to understand the nature of the algorithm and its expected behavior we discuss briefly some basic concepts.

A convex set is defined as follows.

**Definition 4 (Convex Set)** A subset \( C \) of a Hilbert space \( \mathcal{H} \) is said to be convex if together with \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) it also contains \( \mu \mathbf{x}_1 + (1 - \mu)\mathbf{x}_2 \) for all \( 0 \leq \mu \leq 1 \).

Figure 3.7 shows an example of a convex and a non-convex set. We see that Definition 4 corresponds to the intuitive geometric meaning of convexity. A point in our case is an image. We note that the following sets are obviously convex

1. Any linear subspace
2. Band limited functions
3. Space limited functions

Consider a function with a prescribed Fresnel transform magnitude

\[
C = \{ g(\mathbf{x}) \leftrightarrow G(\omega) : |G(\omega)| = M(\omega) \quad \forall \omega \}. \tag{3.32}
\]

Let \( g_1(\mathbf{x}) \) and \( g_2(\mathbf{x}) \) be members of \( C \). Let \( g_3(\mathbf{x}) = \mu g_1(\mathbf{x}) + (1 - \mu)g_2(\mathbf{x}) \) and compute

\[
|G_3(\omega)|^2 = (\mu G_1(\omega) + (1 - \mu)G_2(\omega))(\mu G_1^*(\omega) + (1 - \mu)G_2^*(\omega)) \tag{3.33}
\]

\[
= (1 - 2\mu + 2\mu^2)M^2(\omega) + 2\mu(1 - \mu)\text{Re}\{G_2(\omega)G_1^*(\omega)\} \tag{3.34}
\]
which is not, in general equal to $M^2(\omega)$ for every $0 \leq \mu \leq 1$. The same is clearly true of a constraint on the magnitude of a function.

We define a projection as follows.

**Definition 5 (Projection)** We call $g \triangleq P_1h$ the projection $h$ onto $C_1$ if $g \in C_1$ and

$$
\|g - h\| = \min_{\forall y \in C_1} \|y - h\|
$$

Let us define $P_1$ as an operator that forces the Fresnel transform magnitude constraint to be satisfied. That is,

$$
P_1h = M(\omega)e^{j\phi(\omega)}
$$

where $M(\omega)$ is the prescribed Fresnel transform from Eq. (3.32). In short, $P_1$ replaces the Fresnel transform magnitude with $M(\omega)$. We now show that $P_1$ is a projection operator under the previous definition. We define the inner product in our Hilbert space as

$$
\langle f, y \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx\,dy\,f(x, y)g^*(x, y),
$$

which is the square of the norm. In order to show that $P_1$ defines a projection it must be demonstrated that

$$
\|P_1h - y\|^2 \leq \|y - h\|^2 \quad \forall \ y \neq P_1h, \ y \in C_1.
$$

The Fresnel transform is a unitary or energy preserving linear transform and so has a Parseval’s relation. We may therefore show that

$$
\|G(\omega) - H(\omega)\|^2 \leq \|Y(\omega) - H(\omega)\|^2.
$$

Let

$$
Y(\omega) = M(\omega)e^{j\varphi(\omega)}
$$

$$
H(\omega) = (M(\omega) + N(\omega))e^{j\phi(\omega)}
$$

$$
G(\omega) = M(\omega)e^{j\phi(\omega)}
$$

where $M(\omega) + N(\omega) \geq 0$. Now

$$
\|G(\omega) - H(\omega)\|^2 = N(\omega)^2
$$

$$
\|Y(\omega) - H(\omega)\|^2 = 2M(\omega)[M(\omega) + N(\omega)][1 - \cos(\phi(\omega) - \varphi(\omega))] + N(\omega)^2.
$$

The first term on the right of Eq. (3.40) is always non-negative and thus $P_1$ defines a projection.

We define $C_2$ as

$$
C_2 = \{g(x) : |g(x)| = a(x) \quad \forall x\}
$$

where $a(x) \in \mathcal{R}$ and $a(x) \geq 0 \quad \forall x$. We define $P_2$ by

$$
P_2h = a(x)e^{j\phi_h(x)}
$$

where $\phi_h(x)$ is the phase of $h(x)$. We must show that

$$
\|g(x) - h(x)\|^2 \leq \|y(x) - h(x)\|^2
$$

Let

$$
y(x) = a(x)e^{j\varphi(\omega)}
$$

$$
h(x) = (a(x) + b(x))e^{j\phi(\omega)}
$$

$$
g(x) = a(x)e^{j\phi(\omega)}
$$
where \( a(x) + b(x) \geq 0 \). Now

\[
\|g(x) - h(x)\|^2 = b(x)^2 \\
\|y(x) - h(x)\|^2 = 2a(x)(a(x) + b(x))(1 - \cos(\phi(x) - \varphi(x))) + b(x)^2.
\]

The first term on the right of Eq. (3.45) is always non-negative and thus \( \mathbb{P}_2 \) defines a projection.

We now have two projection operators \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \). Both of the sets are non-convex. In both cases the phase and magnitude of the function \( g_1 = \mathbb{P}_1 h \) or \( g_2 = \mathbb{P}_2 h \) define a unique element of their respective sets. We seek a solution to the fixed point equation

\[
f = \mathbb{P}_1 \mathbb{P}_2 f,
\]

were \( f \) is the desired complex optical amplitude distribution over Plane \( p_1 \). The phase of \( f \) then provides part the the needed set of Radon transforms that define the instantaneous three-dimensional pressure distribution in the ultrasound pulse.

We cannot guarantee the convergence of an iterative restoration scheme based on Eq. (3.46) when the sets corresponding to the projection operators are non-convex. However, it has been shown that they may possess a useful property known as set distance reduction. In order to state this property we define the distance of a point from a set and the Summed Distance Error (SDE) as follows.

**Definition 6 (Distance)** The distance of a point \( h \) from a closed set \( C_i \), denoted by \( d(h,C_i) \) is

\[
d(h,C_i) = \min_{y \in C_i} \|y - h\|.
\]

**Definition 7 (Summed Distance Error)** Let \( C_1, C_2 \) be any two closed sets with projection operators \( \mathbb{P}_1, \mathbb{P}_2 \), respectively. Let \( f_n \) be the estimate of \( f \) at the \( n \)th iteration of the equation

\[
f_{n+1} = \mathbb{P}_1 \mathbb{P}_2 f_n, \quad f_0 \text{ arbitrary}.
\]

Then the performance measure at \( f_n \), denoted by \( J_e(f_n) \) is the sum of the distances between the points \( f_n \) and the sets \( C_1 \) and \( C_2 \). Thus the performance is measured by

\[
J_e(f_n) \triangleq \|\mathbb{P}_1 f_n - f_n\| + \|\mathbb{P}_2 f_n - f_n\|
\]

The quantity \( J_e(f_n) \) is referred to as the Summed Distance Error (SDE).

Note that

\[
J_e(\mathbb{P}_2 f_n) = \|\mathbb{P}_1 \mathbb{P}_2 f_n - \mathbb{P}_2 f_n\| + \|\mathbb{P}_2 f_n - \mathbb{P}_2 f_n\|
\]

where we have used \( \mathbb{P}_2 \mathbb{P}_2 h = \mathbb{P}_2 h \).

We now state a theorem describing an important property of a two-projection iterative restoration scheme [149,168]

**Theorem 4 (Set Distance Reduction)** The recursion

\[
f_{n+1} = \mathbb{P}_1 \mathbb{P}_2 f_n, \quad f_0 \text{ arbitrary}
\]

has the property

\[
J_e(f_{n+1}) \leq J_e(\mathbb{P}_2 f_n) \leq J_e(f_n)
\]

Thus we expect the Summed Distance Error (SDE) to be monotonically decreasing with iteration number. In practice, for the type of objects we are using the algorithm appears to converge. However, we briefly mention the possible difficulty of traps and tunnels.
Figure 3.8: (a) An example of a trap. The solution process begins with a member of set $C_1$. This possible solution point is projected normally onto set $C_2$. The result is then projected normally back onto set $C_1$ to obtain an updated version of the possible solution point. The trap (indicated by the dashed vertical line) represents a solution to the fixed point equation $P_1P_2...P_m f_n = f_n$ where $f_n$ is the result of the $n$th iteration and $P_1,P_2$ are projection operators onto sets $C_1,C_2$ respectively. Traps never exist in a problem where all sets are convex. (b) An example of a tunnel. The relationship between the geometries of sets $C_1$ and $C_2$ causes a marked slowing of convergence.

**Definition 8 (Convergence Traps and Tunnels)** A trap is a fixed point of the composition operator $P_1P_2...P_m$ which is not a fixed point of every individual $P_i$, for $i = 1,2,...,m$, i.e., a point which fails to satisfy one or more of the a priori constraints yet satisfies

$$f_{n+1} = P_1P_2...P_m f_n = f_n$$

(3.48)

If Eq. (3.48) is almost satisfied we have a tunnel.

Figure 3.8 depicts a trap and a tunnel. Convex sets allow tunnels but never allow traps. It is clear that these can completely stagnate (trap) or severely slow progress toward a solution. For the sound fields we have imaged there appears to be little difficulty with traps assuming a good initial estimate of the phase. Slowing of convergence near the solution does seem to occur, however.

**Algorithm Initialization: The Gabor Hologram**

While the property stated in Theorem 4 indicates that the starting point may be arbitrary, in practice much advantage is gained by having a good initial estimate of the phase (and much lost by having a very poor one). An initial estimate of the phase may be obtained by viewing the recorded intensity in Plane $p_2$ as a Gabor hologram. Phase retrieval has been used as a means of dealing with the twin image of Gabor holography [169]. If the phase of a reconstructed Gabor hologram is not severely distorted it may be profitably used as an initialization of the algorithm. While there are other ways of obtaining an estimate (in our case an initial phase of zero appears to work as well) using the phase of a reconstructed Gabor hologram appears to work well for some reconstructions.

Consider a highly transmissive object (such as an ultrasound pulse). Let the object transmittance be given by

$$T(x,y) = t_o + \Delta t(x,y) \text{ where } |\Delta t(x,y)| \ll t_o \ \forall x,y.$$  

(3.49)

That is, assume the optical wave undergoes very weak scattering so that the optical field immediately after passing through the ultrasound pulse is given by $T(x,y) \exp(jkz)$. Let $a(x,y)$ be the Fresnel transform of $\Delta t(x,y)$ and use the fact that plane waves are invariant under propagation to arrive at the following expression for intensity of the optical field after propagating a distance $\ell$ parallel to the $z$-axis.

$$I(x,y) = |t_o + a(x,y)|^2$$

$$= t_o^2 + |a(x,y)|^2 + t_o a(x,y) + t_o a^*(x,y)$$

(3.50)
The Fourier Slice Theorem

Figure 3.9: Depiction of experiment gathering tomographic data. (a) In the experiment measurements linearly proportional to the ray sum through the object as a function of angle are made. In this figure the angle is measured from the positive $r_2$-axis so that the initial measurement for $\theta = 0$ represents the ray sum of the object taken parallel to the $r_1$-axis. (b) The one-dimensional Fourier transform of the projection for $\theta = 0$ is equal to the two-dimensional transform of the object distribution along the $f_2$-axis, where $r_{1,2}$ and $f_{1,2}$ are corresponding spatial position and frequency domain variables. The Fourier Rotation Theorem explains that rotation of the object in the spatial position domain rotates the Fourier transform of the object through an identical angle in the spatial frequency domain. A complete set of measurements of the ray sum of the object distribution over a full 180 degrees of rotation thus yields enough information to determine the two-dimensional Fourier transform of the object and hence the object itself.

where we have taken the observation plane as our phase reference. The intensity $I(x,y)$ in Eq. (3.50) is called the Gabor hologram of the object transmittance in Eq. (3.49). Physically it would be reconstructed by perpendicular illumination of a mask with transmittance linearly proportional to $I(x,y)$ with a plane wave. The reconstructed hologram is observed a distance $\ell$ from the mask (over a plane parallel to the mask). If $|a(x,y)|^2$ is very small, it may be neglected giving a plane wave and two image terms in the observation plane. One of the image terms is the result of propagating $a^*(x,y)$ a distance $\ell$. This is the wavefront we desired to record and reconstruct. The final term is $a(x,y)$ propagated over a distance $\ell$. This, of course, corresponds to the original distribution propagated a distance $2\ell$ and is therefore a distorting signal. If this distorting signal does not too severely corrupt the phase of the reconstructed wave field then it provides a good starting point for the phase retrieval algorithm. It is important to remember, however, that a badly distorted initial phase can adversely affect the performance of the algorithm.

3.4 The Fourier Slice Theorem

Consider the experiment shown in Figure 3.9. For each angle $\theta$ a ray sum $p_\theta(r_2)$ through the object distribution $o(r)$ is obtained. Let us examine the relationship between the projection and distribution of the unrotated object by computing their Fourier transforms. The one-dimensional Fourier transform of the projection is given by

$$P_\theta(f_2) = \int_{-\infty}^{\infty} dr_2 \ e^{-j2\pi f_2 r_2} p_\theta(r_2).$$

The two-dimensional Fourier transform of the object distribution is

$$O(f) = \int_{-\infty}^{\infty} dr \ e^{-j2\pi f r} o(r).$$
We observe that $P_0(f_2)$ is a slice of $O(f)$ along the $f_2$-axis. That is,

$$O(f)\bigg|_{f_1=0} = P_0(f_2).$$  \hspace{1cm} (3.51)

Let $\Theta$ be a rotation matrix such that the product $\Theta r$ rotates the point $r$ about the origin by an angle $\theta$. Then the two-dimensional Fourier transform $O_\Theta(r)$ of the rotated object $o(\Theta r)$ is

$$O_\Theta(r) = \int_{-\infty}^{\infty} dr' e^{-j2\pi r' \Theta^{-1} r} o(\Theta r).$$  \hspace{1cm} (3.52)

Rotations are area preserving transformations making the Jacobian for the change of variable $r' = \Theta r$ unity. Applying this in Eq. (3.52) yields

$$O_\Theta(r) = \int_{-\infty}^{\infty} dr' e^{-j2\pi r' \Theta^{-1} r} o(\Theta r)$$
$$= \int_{-\infty}^{\infty} dr' e^{-j2\pi r' \Theta \Theta^{-1} r} o(\Theta r')$$
$$= O(\Theta f).$$  \hspace{1cm} (3.53)

Equations (3.51) and (3.4) allow us to obtain the two-dimensional Fourier transform of an object by recording a complete set of projections for $0 < \theta < \pi$. This set of projections is referred to as a Radon transform. It is clearly invertible as the original object distribution can be obtained from it via the Fourier transform. In practice, however, only a finite set of points $\{r_{2,k}, \theta_k\}$ are measured. Sampled Radon transforms are not, in general, invertible. Sampling is not one-to-one. Shannon’s sampling theorem explains how two different vectors (sinusoidal signals) are mapped to the same coefficient through aliasing or “spectral folding”. Consider a complete sinusoidal basis for a complex vector space and a continuous function. The sampling operator records some sinusoidal basis vectors properly (those whose frequency is strictly below half the sampling rate), some not at all (those harmonics with zeros falling on the sampling points) forming the nullspace of the operator, and some incorrectly. In the latter case, the coefficients of the high harmonics will be added to or “folded over” onto the coefficients of lower harmonics. The key to inversion, or at least, approximate inversion is to only sample a subclass or subspace of all possible functions. Specifically those whose nonzero coefficients are all attached to basis functions we can record in an invertible fashion. This eliminates functions with higher frequency components, some of which form the nullspace of the sampling operator. In practice, adequate sampling takes place at twice the frequency of the lowest frequency signal we are willing to alias.

It is clear from the above development of the Fourier Slice theorem that sampling in angle occurs in the Fourier domain. That is, the density of angular samples must be based on the frequencies present in the Fourier domain representation of the signal. The polar representation of the continuous Fourier transform of the object $O(r, \theta)$ is $2\pi$-periodic and thus admits a Fourier series expansion

$$O(r, \theta) = \sum_{n=-\infty}^{\infty} c_n(r)e^{-jn\theta}.$$  \hspace{1cm} (3.54)

For a given $r$, as we move along the $\theta$-coordinate the function possesses harmonics $\exp(-jn\theta)$. The number samples along this function per $2\pi$ period is the total number of angular views, $p$. If there are $p$ views per $2\pi$ period, the first harmonic inaccurately represented is the one having $(p-1)/2$ cycles per $2\pi$ period. Let us assume $p$ is odd so that we may talk of minimal sampling. If $p$ is odd the $(p-1)/2$ harmonic is adequately sampled. Thus $n$ in Eq. (3.54) must be limited to $(p-1)/2$ requiring that the object have a Fourier series representation of

$$O(r, \theta) = \sum_{n=-(p-1)/2}^{(p-1)/2} c_n(r)e^{-jn\theta}.$$
The Fourier Slice Theorem

Figure 3.10: (a) A slice normal to the axis of rotation used for tomographic data collection. The sinusoidal wave pattern present in the image undergoes significant changes in direction as the slice is rotated in the plane of the page. (b) Its two-dimensional Fourier transform has significant energy in higher harmonics within the nullspace of the angular sampling operator. These frequencies will not be recorded, leading to error in the reconstruction.

An object that is radially symmetric about the axis of rotation will have a radially symmetric Fourier transform (no higher harmonics in the angular Fourier series expansion of the Fourier transform), so that angular sampling rates will increase with increasing angular asymmetry of the spatial domain signal.

Having determined the linear and angular sampling rates we briefly discuss a difficulty arising with the tomographic measurement of arbitrary sound fields under the angular bandwidth constraint. Sinusoidal wave patterns oriented so that their direction changes significantly with projection angle produce high angular harmonics close to the nullspace of the angular sampling operator. An example of such a pattern is given in Figure 3.10(a). Figure 3.10(b) shows the two-dimensional Fourier transform of the object in Figure 3.10(a) superimposed on a possible sampling grid. Most of the energy in the signal is not sampled, leading to significant errors in the reconstruction.

In applying sampling criteria to the measurement experiment it must be remembered that intensity is the actual physical quantity sampled. It is necessary to relate the frequencies present in the phase function to those present in the optical intensity pattern. Sec. 2.5 shows that under weak scattering assumptions it is approximately correct to consider the bandwidth of the phase function and the bandwidth of the optical signal to be the same. The bandwidth of the optical intensity will be twice that of the complex optical amplitude. Thus the projection sampling rate computed from the optical phase function is multiplied by a factor of two to obtain the correct sampling rate for the optical intensity in the experiment. The angular sampling rate computed above applies directly.
In order to more completely understand the nature of the physics and signal processing involved in the measurement technique described in this thesis, a simulation of the complete experiment and reconstruction process was computed. Figure 4.1 depicts the measurement of a single view of the field. Incident from the right are planar optical wavefronts (see Figure 4.1(a)). In the diagram of the physical experiment in Figure 2.1 these are produced by the laser source and collimating lens. The output of the transducer in Figure 4.1(e) is simulated via a Fresnel model for a simple acoustic aperture function. The accumulated (in the direction normal to the optical wavefronts) optical refractive index is pictured beneath the transducer aperture in Figure 4.1. The data in the central vertical column of this image are plotted in Figure 4.2(a). In Figure 4.1(c) is a representation of the optical phase or optical wavefront shape indicating the accumulated delays and advances experienced by the wavefront as it passed through the ultrasound pulse. Under our weak scattering model, points on this surface plot deviate from zero in direct proportion to the ray sum of the refractive index variation induced by the acoustic disturbance. The magnitude of the optical wave over this plane is approximately uniform. The location of Figure 4.1(c) corresponds to Plane $p_1$ in Figure 2.1. Figure 4.1(d) depicts the optical intensity that is imaged onto the Charge Coupled Device (CCD) array. The data in the center column of the image are plotted in Figure 4.2(d). Plane $p_2$ is the corresponding location in Figure 2.1. Lenses $\ell_2$ and $\ell_3$ in Figure 2.1 image this plane onto the CCD array. An interesting side note is that the intensity data may possess some similarity to the phase data if the plane in which they are measured is not too far from the pure phase delay plane. This may be understood by recognizing that the ultrasound pulse induces only a small broadening of the optical angular plane wave spectrum. Prior to passage through the ultrasound pulse the angular plane wave spectrum of the optical wave (neglecting finite aperture effects) consists of a Dirac delta function at the center of frequency space. As a crude approximation we may consider the broadening induced by the scattering event to add a symmetric pair of Dirac delta functions representing the center frequency of the transducer. Using a Fresnel propagation model to move this spatial Frequency representation from the transducer plane (the plane immediately subsequent to the scattering event) to the diffraction plane (the plane imaged onto the CCD array) we multiply the symmetric pair of Dirac delta functions by a constant according to Eq. (2.31). Assuming a low amplitude acoustic disturbance we use Taylor’s approximation to obtain

$$u(r_\perp) = e^{i\phi(r_\perp)} \approx 1 + i\phi(r_\perp)$$

where $\phi(r_\perp)$ represents the delays and advances induced in the optical wave front by passage through points whose density and hence refractive index are different from ambient values. The resultant intensity in the diffraction plane is thus

$$I(r_\perp) = |1 + ip\phi(r_\perp)|^2 = 1 - 2\text{Im}(p)\phi(r_\perp) + |p|^2\phi^2(r_\perp)$$
Figure 4.1: A schematic of the simulated experiment. (a) Planar optical wavefronts propagating toward the two-dimensional phase delay function representing the ultrasound in (b). In the actual experiment these are produced with the laser and collimating lens in Figure 2.1. (b) The ray sum (taken in the direction of the optical wave propagation) of the ultrasound field. The field is produced by the transducer in (e). (c) Optical wavefronts immediately after passing through the ultrasound field. The amplitude remains essentially unchanged. The phase delay is proportional to the “projection” (in the direction of optical wave propagation) of the ultrasound field. Their location corresponds to Plane $p_1$ in Figure 2.1. (d) Optical intensity after propagating a known distance beyond interaction with the ultrasound field. In Figure 2.1 this location is specified by Plane $p_2$. (e) The ultrasound transducer producing the field in (b). In the simulation a Fresnel model for acoustic propagation is used. The method described in Appendix B is used to compute the projection or ray sum of the three-dimensional field.
where \( p \) represents the Fresnel Fourier domain propagator in Eq. (2.31) evaluated at the location of the symmetric Dirac delta function pair. Equation (4.2) indicates that for some values of \( p \) (controlled by the location of the diffraction plane and the pulse center frequency) and reasonably small amplitudes in the ultrasound pulse the diffraction plane image will have similarities to the delay function we seek.

The simulation used 4 cycles at a center frequency of 1.25 MHz. The time series was filtered to approximately represent a transducer with 80 percent bandwidth. A D-shaped aperture was driven and a Fresnel model used to propagate the pulse 40 acoustic wavelengths from the aperture. It would perhaps be more appropriate to use a full Huygens model for the propagation of the acoustic pulse. However, the general features of the pulse are adequate for simulation purposes. In order to effect this computation the D-shaped aperture was sampled, filtered and subsampled to reduce the introduction of aperture aliasing effects. Its projection in the direction of optical wave propagation was computed and then the propagation was done in two-dimensions (see Appendix B). The two-dimensional distribution was used to create a phase perturbation on a planar optical wave with maximum deviation from zero \( \pi/5 \). A Fresnel model was then used to propagate this distribution 8 cm. The squared modulus in this plane represents the experimentally measured optical intensity. Figures 4.2(a) and (b) show the simulated optical phase delay and diffraction plane intensity respectively. In order to simulate experimental conditions more fully the first diffraction plane intensity image in the data set discussed in Sec. 5.3.2 was used to estimate a signal peak to noise variance ratio (under the assumption that the measurement noise may be represented by a Gaussian white noise random field) of \( 8.6 \times 10^{-6} \). Zero mean Gaussian noise was then added to each simulated intensity image creating the same apparent signal peak to noise variance ratio present in the data. A total of sixty views, evenly spaced over 180 degrees were taken for reconstruction.

Several views of the phase phantom are shown in Figure 4.3. One hundred thirty iterations per view were used to retrieve the phase from the noisy intensity plane images. Figure 4.6 shows the phase retrieved image corresponding to the first view. A typical difficulty evident in both line plots is the reduction in amplitude of the higher peaks in the phase image. Figure 4.6(d) shows the monotonically decreasing Summed Distance Error (SDE). Under the weak scattering model each row of a phase retrieved view belongs to a separate Radon transform of a level slice of the ultrasound pulse. As we move from row to row in an image (rows are oriented approximately perpendicular to the direction of acoustic propagation) we move to a different level slice. The complete set of phase retrieved views thus provides a set of Radon transforms (one for each row or level in a given image) allowing complete reconstruction the instantaneous refractive index distribution in the pulse. Figure 4.7 shows the Radon transform of the pulse at several levels. Figure 4.8 shows the results of applying the Fourier Slice Reconstruction (FSR) algorithm to two levels. Figures 4.8(a) and (d) show the reconstructed pressure distribution approximately one quarter and one half of the height of a view from the transducer respectively. Figures 4.8(b) and (e) show the corresponding phantom pressure distributions. Line plots are shown in Figures 4.8(c) and (f). Figure 4.8 shows a comparison that includes points that are atypically poorly matched. This is a result of the fact that the height of the higher peaks in the phase image is retrieved more slowly than the finer details. Also, as seen in the image in Figure 4.8(d), the line lies on a relatively steep gradient in the vertical direction thus emphasizing the differences between the functions.
Figure 4.2: (a) The phase phantom (the optical phase in Plane $p_1$ of Figure 2.1 and in Figure 4.1(c)). (b) The computed optical intensity in Plane $p_2$ of Figure 2.1 and in Figure 4.1(d). (c) A line plot of the data under the vertical line in (a). (d) A line plot of the data under the vertical line in (b).
Figure 4.3: Several views of the phase phantom evenly spaced in angle over 180 degrees. The view order begins at the upper left image and proceeds left to right.
**Figure 4.4**: Several views of the simulated diffraction plane intensity evenly spaced in angle over 180 degrees. These correspond to the phase phantoms shown in Figure 4.3.
Figure 4.5: A comparison of phantom and retrieved phase for increasing iteration number in the phase retrieval algorithm. The solid line plots the phantom phase. (a) The retrieved phase for the first view in the simulated experiment. (b) A line plot showing the retrieved phase after 30 iterations of the phase retrieval algorithm. (c) 60 iterations. (d) 90 iterations. (e) 120 iterations. (f) 150 iterations.
Figure 4.6: (a) The retrieved phase for the first view in the simulated experiment. (b) Line plot for the central vertical line in Figure 4.6(a). The solid line plots the phantom phase. (c) Line plot for the leftmost vertical line in Figure 4.6(a). The solid line plots the phantom phase. (d) The Summed Distance Error (SDE) for the one hundred thirty iterations in the phase retrieval.
Figure 4.7: The Radon transform for several levels in the simulated ultrasound pulse. The upper left image is for a level near the transducer aperture. Moving left to right moves to levels further away from the aperture. In each image the horizontal coordinate is view number (or angle) and the vertical is sensor number. The levels are evenly spaced from the top (nearest the transducer aperture) to the bottom (furthest from the transducer aperture) of the view volume. The scale in each image is linear. However, different scales have been used for each view to cover the wide dynamic range of the data.
Figure 4.8: (a) Reconstruction of a level parallel to and near the transducer aperture. Data under the vertical line are plotted against corresponding data from the original phantom (subfigure (b)) in (c). (b) Phantom data corresponding to reconstruction in (a). (c) Line plot comparison between reconstructed and phantom data in (a) and (b) respectively. (d) Reconstruction of a level parallel to the transducer aperture and corresponding to the central row in the views of Figure 4.6. Data under the horizontal line are plotted against corresponding data from the original phantom (subfigure (e)) in (f). (d) Phantom data corresponding to reconstruction in (c). (f) Line plot comparison between reconstructed and phantom data in (d) and (e). This is an example of an atypically poor match between the phantom and reconstruction. High frequency components of the phase are generally recovered more slowly than low frequency components. This is perhaps evidence of a tunnel (see Figure 3.8). Also, as seen in the image in (d), the line lies on a relatively steep gradient in the vertical direction thus emphasizing the differences between the functions.
Basic calibration of the experiment, demonstration of the linearity of the optical measurement process, and investigation of the performance of the background removal algorithm are discussed in this chapter. Calibration of the physical system and investigation of the linearity of the instrumentation used to record optical intensity are covered in Sec. 5.1. Linearity of the measurement process itself is examined in Secs. 5.2.1 and 5.2.2 by explicit experimental testing of the superposition and scaling properties satisfied by linear systems. Sections 5.3.1 and 5.3.2 demonstrate the phase retrieval and reconstruction process. The latter examines the effectiveness of a wavelet-based filter (see Sec. 3.2.2) in removing background corruption that varies slowly from view to view as the ultrasound field is rotated about an axis normal to the propagation direction of the incident optical pulse during the data collection process.

5.1 System Calibration

An understanding of the nature of the Charge Coupled Device (CCD) array and Image Acquisition (IMAC) board mappings is required in order to make explicit the relationship between the physics of the experiment and the measured data. Measurements of optical intensity must be linearly related to the actual optical intensity in the measurement plane in the imaging volume. Hence we look at the linearity of the CCD camera as a recording device. The CCD camera produces an analog NTSC video signal which undergoes analog to digital conversion via an IMAC board. This conversion transforms the signal via an approximately affine mapping which must be inverted in order to maintain the necessary linear relationship between the discrete digitized samples in the computer and the continuous analog physics of the experiment.

5.1.1 CCD Camera Linearity

If a minimum reference voltage of zero volts is used on the IMAC board then the mapping is a linear (as opposed to affine) one. We then vary the number of laser pulses per field to establish a mapping for the CCD array. After generating a series of images each using a different number of laser pulses per field we may plot the result as in Figure 5.1 for a single element of the array. It is clearly very linear over a large range. Typically, four laser pulses per field are used to record a view.

5.1.2 Digitization of Optical Intensity Data

It is not typically desirable to use settings on the IMAC board that place the minimum voltage at zero. Doing so may severely limit the usable output range of the data. The values in Table 5.1 depict
Figure 5.1: A plot depicting the nature of the CCD integration. The horizontal axis specifies the number of laser pulses per field. The vertical axis indicates the measurement of optical intensity after the affine transform implemented by the IMAC has been inverted and the darkfield image (image returned by the CCD when no photons impinge on the wells) subtracted. The intensity measurement is clearly linear over a large range.

Figure 5.2: (a) CCD darkfield image for a subregion of the CCD array. (b) A line plot of the data under the vertical line in (a).
System Calibration

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^aStandard image

Table 5.1: Data for the calibration of the Image Acquisition (IMAC) board used to digitize the optical intensity data in the National Television Standards Committee (NTSC) video signal.

how effective the IMAC mapping inversion is at making the board a linear recording device. In order to generate these values a series of background images was taken with various IMAC board settings and laser exposures (see Sec. 3.1). Each image underwent the inverse IMAC mapping, background division and was normalized according to the number of laser pulses per field used to record the image. The ratio \( r \) of the image to a “standard” image (taken with the IMAC board settings shown in Table 5.1) is estimated as the mean of the point-by-point ratio of the two images (the images do not have values near zero) or

\[
r = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \frac{I(m,n)}{I_s(m,n)},
\]

where the test and standard images \( I(m,n) \) and \( I_s(m,n) \) are \( M \times N \). Table 5.1 shows the results. It is clear that for certain settings of the reference scale factor the inverse mapping used is insufficient to yield a linear relationship.

5.1.3 Imaging Plane Location

Application of the phase retrieval algorithm requires knowledge of the physical distance between the two planes over which the magnitudes are measured (Planes \( p_1 \) and \( p_2 \) in Figures 2.1 and 3.2). In order to make this measurement as precise and convenient as possible the system was calibrated in the following manner.
A set of data matching position of the image plane with the rotational position of the camera lens was taken. The rotational position of the camera lens was assessed by a vernier scale attached to the lens itself. The location of the image plane was moved by physically moving the location of the transducer in the imaging volume a known distance along a path approximately parallel to the system optical path. These data allow fitting of the system imaging parameters to our simple optical system model so the location of the plane current image plane can be calculated instead of having to measure it in some explicit fashion each time. Figure 5.3 shows the model fit along with some calculated values for system magnification.

We may verify the reasonableness of the magnification by using a phase retrieved image, knowledge of the center frequency of the pulse being measured, and an approximation for the speed of sound in water to compute the size of one dimension of a single CCD imaging cell. For the data set discussed in Sec. 5.3.2 using the intensity data for which Plane $p_2$ in Figure 2.1 is located furthest from Plane $p_1$, the distance between Planes $p_1$ and $p_2$ and the image magnification factor were estimated at 9.37 cm and 5.91 respectively. Using a wave speed of 1440 m/s and zero crossings of the main pulse (the center frequency was 2.25 MHz) it was estimated that the center to center spacing of the CCD array elements is 12 microns. The service manual for this particular CCD array gives the specification as 13 microns [170].

## 5.2 Measurement Linearity

A linear measurement operator $\mathcal{M}$ satisfies the scaling and superposition properties in Eq. (5.2).

$$\mathcal{M}\{as_1(r_\perp) + bs_2(r_\perp)\} = a\mathcal{M}\{s_1(r_\perp)\} + b\mathcal{M}\{s_2(r_\perp)\}$$

(5.2)

The ability of this method to provide quantitative measurements of sound fields was tested explicitly in the following two experiments.

---

1. The intensity was recorded over the full 180 degrees in three planes. One of these was the plane in which the transducer sits (Plane $p_1$ in Figure 2.1). The other two were diffraction planes further along the optical axis in the direction of optical wave propagation.

2. Actually, the service manual indicates that the cell size is 11 x 13 microns in the horizontal and vertical directions respectively. The vertical direction indexes from line to line through field 0 and field 1 of the interlaced readout. Thus the horizontal direction is converted to an analog signal as part of the National Television Standards Committee (NTSC) video specification. The Image Acquisition (IMAC) board then digitizes this direction at a spacing of 13 microns to produce “square” pixels.
Table 5.2: Data for determining the relationship between the rotational position of the camera lens and the location of the image plane.

5.2.1 Superposition of Refractive Index Perturbations

Three different driving voltage waveforms \( w_1 \), \( w_2 \), and \( w_3 = w_1 + w_2 \) were used to drive a Panametrics, 12.5 mm diameter, 2.25 MHz ultrasound transducer. The voltage waveform \( w_1 \) was 2.22 microseconds of a 2.5 MHz tone burst. Voltage \( w_2 \) was 2.22 microseconds of a 2 MHz tone burst. In each case, foreground (sound on) and background (sound off) images in the transducer plane and a diffraction plane were taken. After mapping to values proportional to the actual diffraction plane intensity, the phase of the optical pulse immediately after passage through the sound field was recovered. The sum of the phase delay produced using voltage waveforms \( w_1 \) and \( w_2 \) is plotted with that produced by driving voltage \( w_3 \) in Figure 5.4. A very good agreement is apparent.

5.2.2 Amplitude Scaling of Refractive Index Perturbation

To test the amplitude scaling property of the measurement operator several linearly scaled versions of the same driving waveform (five cycles at 2.25 MHz) were applied to the transducer via a 46 db amplifier. The signal voltages (before amplification) ranged from 10 to 175 mV in increments of five millivolts. In each case foreground (sound on) and background (sound off) images from the transducer and diffraction planes were used to compute the phase delay introduced into the optical wavefront by the ultrasound pulse. Figure 5.5(b) shows line plots of the some of the phase waveforms corresponding to an evenly spaced subset of the driving voltages. Figure 5.6 is a more quantitative statement on the scalar multiplication property. The ratio for each possible pairing \( (P_1, P_2) \) of the 34 driving voltages was estimated in four different manners. For Figure 5.6(a) the image ratio was calculated as

\[
r = \frac{1}{N} \sum_{m,n \in S_N} \frac{v_2 P_1(m,n)}{v_1 P_2(m,n)},
\]

where \( S_N \) is the set of all pixel values that deviate from zero by more than a noise threshold, \( N \) is the number of elements in the set, and \( v_i \) is the driving voltage for the transducer. The noise threshold used in this analysis is four times the maximum pixel value in a region of the recovered phase image.
Figure 5.4: The above plots demonstrate the compliance of the measurement operator to the requirements of superposition. Images in subfigures (a)–(d) are 34 mm in the vertical and 14.2 mm in the horizontal dimension respectively. (a) Phase retrieved image of the waveform produced by driving voltage $w_1$. (b) Phase retrieved image of the waveform produced by driving voltage $w_2$. (c) Phase retrieved image of the waveform produced when the transducer is driven with the algebraic sum of waveforms $w_1$ and $w_2$. (d) Sum of images in (a) and (b). Figures 5.4(e) and (f) show line plots comparing the images in Figures 5.4(c) and (d). The solid line plots data from (c).
Background Correction

where no signal is present. The maximum pixel value was obtained from the image produced with the lowest driving voltage. Figure 5.6(b) shows the product of the ratio of image maxima and the inverse driving voltage ratio. That is, for an image pairing \((P_1, P_2)\), we have

\[
r = \frac{v_2 \max(P_1(m, n))}{v_1 \max(P_2(m, n))}
\]  

(5.4)

Figure 5.6(c) shows the product of the ratio of image minima and the inverse driving voltage ratio. That is, for an image pairing \((P_1, P_2)\), we have

\[
r = \frac{v_2 \min(P_1(m, n))}{v_1 \min(P_2(m, n))}
\]  

(5.5)

Figure 5.6(d) shows the product of the ratio of image range and the inverse driving voltage ratio. That is, for an image pairing \((P_1, P_2)\), we have

\[
r = \frac{v_2[\max(P_1(m, n)) - \min(P_1(m, n))]}{v_1[\max(P_2(m, n)) - \min(P_2(m, n))],}
\]  

(5.6)

From Figure 5.5(c) reasonably good linearity is apparent. The linearity is excellent for voltages ratios that do not involve higher ultrasound pressures. It should be noted that the higher ratios always involve a high pressure and a low pressure measurement thus emphasizing the difficulty with high positive peaks. This difficulty with higher pressures is not present in the simulations and may result from a measurement error, internal scattering in the sound field, or the nonlinear relationship between pressure and density.

5.3 Background Correction

Sections 5.3.1 and 5.3.2 investigate the background correction methods used in the remainder of this thesis. Section 5.3.1 presents a simple phase retrieval after performing the basic background correction describe in Sec. 3.2.1. Section 5.3.2 compares a three-dimensional reconstruction of pressure obtained from experimental data using the background correction in Secs. 3.2.1 and 3.2.2 with a reconstruction of the same data using only the simple method described in Sec. 3.2.1. The wavelet-based filtering algorithm contributes substantially to the reduction of background corruption.

5.3.1 Simple Phase Retrieval

This section describes a simple phase retrieval and background correction. Figure 5.7(a) shows a diffraction plane image without background correction as discussed in Sec. 3.2. Clearly evident in the image are Airy patterns from dust particles or point imperfections in the optical elements. The image aspect ratio is distorted to make some of the structure in the diffraction pattern from the acoustic pulse more apparent (thus explaining why the Airy patterns are not circular). Also seen, are the patterns resulting from multiple reflections in the optical system. Figure 5.7(b) shows the same data after background correction. The line plots in Figures 5.7(c) and (d) correspond to the data under the vertical and horizontal lines in Figure 5.7(b) respectively.

The image in Figure 5.8(a) contains data proportional to the diffraction plane intensity \(9.37\) cm from the transducer plane. The aperture dimensions are \(34.0\) mm \(\times\) \(12.8\) mm (Horizontal \(\times\) Vertical). Figure 5.8(b) shows the transducer plane phase after 40 iterations of the phase retrieval algorithm. Figures 5.8(c) and (d) show line plots of the diffraction plane intensity and transducer plane phase respectively.

5.3.2 Tomographic Reconstruction: Circular Aperture

Here we present a full tomographic reconstruction of the field from a Panametrics 2.25 MHz, 12.5 mm diameter, unfocused transducer. The transducer was driven with six cycles of its center frequency
Figure 5.5: (a) Reference image for line plots in (b). (b) The plot on the right side shows the total measured optical delay for various evenly spaced scalar multiples of the same transducer driving waveform. Typical of the experimental data obtained for this thesis is the tendency for the computed optical wavefront to have positive peaks that are slightly lower than expected at higher ultrasound pressures. This may be seen in the line plots shown here. It may be a result of the nonlinear relationship between pressure and density.
Figure 5.6: Each of the plots (a)–(d) shows a measure of linearity. In each case the linearity measure is plotted against voltage ratio. Reasonable linearity is observed.  
(a) A line plot showing the ratio of two images computed according to Eq. (5.3).  
(b) A line plot showing the ratio of maximum positive peaks in two images as a function of input voltage ratio. Apparent under estimation of positive peaks for higher ultrasound pressures leads to higher than unity linearity measures.  
(c) A line plot showing the ratio of image minima.  
(d) A line plot showing ratio of image ranges (maximum - minimum).
Figure 5.7: (a) A raw (not background corrected) view of the diffraction plane pattern produced by a Panametrics 2.25 MHz, 12.5 mm diameter unfocused transducer driven by six cycles at the center frequency. Fifty images were averaged to produce this view. (b) The data in (a) after background division. (c) Line plot comparison of data under the vertical line in (b) with the corresponding data in (a). (d) Line plot comparison of data under the horizontal line in (b) with the corresponding data in (a).
Figure 5.8: (a) Diffraction plane intensity data 9.37 cm from the transducer plane. The aperture dimensions are 34.0 mm vertical and 12.8 mm horizontal. (b) Transducer plane phase after 40 iterations of the phase retrieval algorithm. (c) Line plot of data under horizontal line in (a). (d) Line plot of data under horizontal line in (b).
using a 70 mV amplitude sine wave through a 46 db power amplifier. The pulse was captured approximately 9 $\mu$s into its temporal evolution. Figure 5.9 shows the effect of wavelet filtering to remove corruption resulting from higher frequency error sources that vary slowly as the ultrasound field is rotated during the data collection process (see Sec. 3.2.2). A marked reduction in apparent ridges or streaking is evident. Figures 5.10 and 5.11 show corresponding measured diffraction plane (Plane $p_2$ in Figures 2.1 and 3.2) intensity and retrieved transducer plane (Plane $p_1$ in Figures 2.1 and 3.2) phase obtained via 40 iterations of the phase retrieval algorithm. A total of 101 views spaced 1.8 degrees apart in angle$^3$ were processed for this data set. A diffraction plane separation of 9.37 cm was used.

Figures 5.12 and 5.13 show tomographic reconstructions using phase retrieved data from filtered and unfiltered data respectively. The tomographic center of rotation was determined via convolution between the 0 and 180 degree phase retrieved views. For the figures in Figure 5.12 the wavelet filtering algorithm was applied to levels below 100 out of 180 as only the reconstructions containing higher amplitudes suffer noticeably from ring artifacts. Filtering was performed recursively to 3 levels (zeroing the horizontal detail image at each level). Figure 5.12(a) is a reconstruction of a plane with normal parallel to the tomographic rotational axis (roughly parallel to the surface of the transducer face or the direction of acoustic propagation). The pulse is propagating into the page and most of it has passed the reconstructed plane. We are thus seeing detail in the “backfield” or “ring down” of the transducer. Of note are the modal like symmetries in the field and a vague hint of reconstructed ridge or ring artifact. The incomplete reconstruction of the outer rings is a result of incomplete tomographic data. The edge waves rotated out of the field of view during data acquisition. This particular level corresponds to row 60 out of 180 in Figures 5.12(d) and (e). Figure 5.12(b) shows level 120 out of 180. Somewhat more ridge or ring artifact is evident. This is typical of the reconstructions containing the higher amplitudes. However, in general the noise level is quite low. It should be noted that no filtering other than the wavelet prefiltering for the purpose of ridge corruption removal was performed on any of the data. This should be evident in the line plot in Figure 5.12(f). Level 140 out of 180 is shown in Figure 5.12(c). Again the prefiltering did a good job of removing the ridges resulting from corruption that varies only slowly from view-to-view. The field itself, in this case, possesses circular symmetry and contains rings. However, ridge artifacts discussed in Sec. 3.2.2 always reconstruct as circles or curves centered about the tomographic rotational axis. Figures 5.12(d) and (e) show slices through the reconstructed volumes taken normal to those in Figures 5.12(a), (b), and (c). These planes have normals perpendicular to the tomographic rotational axis and approximately perpendicular to the transducer surface normal. Figure 5.12(d) is near the axis of symmetry for the acoustic pulse and allows a very good view of the edge wave interference pattern. Figure 5.12(e) is off-center, nearer the edge of the pulse. The line plot in Figure 5.12(f) shows data under the vertical line in Figure 5.12(d). An interesting feature is the change in spatial wavelength as we move from left to right in the plot. This happens as the line plot moves between regions of the pulse with local wavefronts oriented at different angles.

Figure 5.13 shows the same views as Figure 5.12 using data that was not filtered to remove ridge corruption. The image in Figure 5.13(a) is, of course, the same as no filtering was used here in either case. However, the images in Figures 5.13(b), (c), and (d) show significant corruption. The off-center view in Figure 5.13(e) is sufficiently far from the tomographic center of rotation that no artifact is apparent. Figure 5.13(f) shows a comparison between the data under the vertical line in Figure 5.12(d) and the corresponding data in Figure 5.13(d).

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$^3$This includes a view at 180 degrees.
Figure 5.9: A comparison of diffraction plane views with and without wavelet prefiltering to remove ridges resulting from corruption that varies very slowly view-to-view (ridge artifacts). The left column is diffraction plane data for different levels (distances from the transducer) of the acoustic pulse with no prefiltering. The horizontal axis indexes view angle. The vertical axis indexes CCD charge well number. The right column shows the same levels after filtering to remove ridge corruption. A different linear gray scale is used at each level to display the wide dynamic range of the data without distorting feature shape.
Figure 5.10: Several views in the diffraction plane. View angles 0 through 144 degrees are depicted beginning with the image in the upper left and progressing down the columns in increments of 18 degrees. The corresponding phase retrieved views are shown in Figure 5.11.
Figure 5.11: Several views of the retrieved phase in the transducer plane. View angles 0 through 144 degrees are depicted beginning with the image in the upper left and progressing down the columns in increments of 18 degrees. A diffraction plane separation of 9.37 cm was used. The corresponding diffraction plane views are shown in Figure 5.10.
Figure 5.12: Figures (a)–(f) show the results of tomographic reconstruction of the field from a circular aperture transducer. The filtering discussed in Sec. 3.2.2 was used to reduce the ring artifact. (a) Level 60 out of 180 approximately parallel to transducer surface. (b) Level 100 out of 180. (c) Level 120 out of 180. (d) A slice approximately normal to the transducer surface near the acoustic pulse’s axis of symmetry. (e) A slice, parallel to that in subfigure (d) closer to the edge of the pulse. (f) A line plot of the data under the vertical line in subfigure (d). A different linear gray scale is used in (a), (b) and (c) to display the wide dynamic range of the data without distorting feature shape. Compare Figure 5.13.
Figure 5.13: Figures (a)–(f) show the results of tomographic reconstruction of the field from a circular aperture transducer without any prefiltering to reduce source of ring artifact. (a) Level 60 out of 180 approximately parallel to transducer surface. (b) Level 100 out of 180. (c) Level 120 out of 180. (d) A slice approximately normal to the transducer surface near the acoustic pulse’s axis of symmetry. (e) A slice, parallel to that in subfigure (d) closer to the edge of the pulse. (f) A line plot of the data in the central vertical column of the image in (c) compared with that in Figure 5.12. A different linear gray scale is used in (a), (b) and (c) to display the wide dynamic range of the data without distorting feature shape. Compare Figure 5.12.
CHAPTER 6

TOMOGRAPHIC RECONSTRUCTION

This chapter presents several data sets demonstrating the phase retrieval, and tomographic reconstruction of various fields. Sections 6.1 and 6.2 respectively cover fields from annular and semi-circular apertures driven with an unfocused 2.25 MHz transducer. Sections 6.3 and 6.4 respectively discuss reconstruction of a wide bandwidth field produced with elements 32 and 34 of a 64 element, 2.5 MHz linear array transducer, and elements 44 and 21 of the same. Full measurements of a wide bandwidth field from a 12.5 mm diameter, 2.25 MHz, unfocused transducer with and without a cube of beef (approximately 150 mm × 150 mm × 150 mm in size) in front of the transducer are presented and discussed in Sec. 6.5.

6.1 Annular Aperture

This data set was created using the transducer from Sec. 5.3.2. A dot (approximately 4 mm in diameter) of absorbing tape was placed near the center of the transducer aperture in order to create a nearly annular field pattern. The transducer was then driven with 6 cycles of its center frequency and the pulse wave imaged approximately 12 µs into its temporal evolution. Wavelet filtering was applied to levels below 65 out of 140. Filtering was performed recursively to 3 levels (zeroing the horizontal detail image at each level). The distance from the diffraction plane (Plane $p_1$ in Figure 2.1) to the transducer plane (Plane $p_2$ in Figure 2.1) is 10.28 cm. Forty iterations of the phase retrieval algorithm were used. The aperture dimensions (the physical dimensions of the image used in the phase retrieval) are 34.0 mm × 9.9 mm.

Figures 6.1 and 6.2 show corresponding diffraction plane intensity and retrieved transducer plane phase. Evidence of the annular nature of the field can be seen in both figures. Radon transforms for several planar slices of the pulse are depicted in Figure 6.3. The corresponding reconstructions are in Figures 6.4 and 6.5. Evidence of inadequate filtering for ridge removal may be seen in the last image of Figure 6.5. The quantitative nature of tomographic imaging allows more than merely the determination of basic structure. The values shown in the various reconstructed images are proportional to the instantaneous refractive index (and therefore pressure in our acoustic propagation model). A surface plot shows these values explicitly for the level corresponding to row 100 in the views of Figure 6.2. Note that the pulse is moving through this plane at an angle.

6.2 Semi-Circular Aperture

Several layers of absorbing tape were used to cover half the aperture of an unfocused, 2.25 MHz, 12.5 mm diameter, Panametrics transducer producing a semi-circular aperture. The transducer was
Figure 6.1: Diffraction plane intensity for the annular aperture experiment. Each view is 34 mm in the horizontal and 9.9 mm in the vertical dimension. These views were taken approximately 10.28 cm from the transducer plane (Plane $p_1$ in Figure 2.1). View angles 0 through 144 degrees are depicted beginning with the image in the upper left and progressing left to right in increments of 18 degrees. Compare Figure 6.2.
Figure 6.2: Optical phase just after interaction with the ultrasound pulse. The phase retrieval algorithm was stopped after 40 iterations. View angles 0 through 144 degrees are depicted beginning with the upper left image and progressing left to right in increments of 18 degrees. Each view is 34 mm in the horizontal and 9.9 mm in the vertical dimension. Compare Figure 6.1.
Figure 6.3: Radon transforms for several planar slices through the ultrasound pulse obtained from the phase retrieved data in the transducer plane. Plane normals are parallel to the tomographic rotational axis and approximately parallel to the transducer surface normal. The levels correspond to rows 15, 30, 45, 60, 75, 90, 105, and 120 in the views in Figure 6.2 beginning with the upper left figure and progressing left to right. Note the incomplete removal of ridges in level 120. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. Compare Figures 6.4 and 6.5.
Figure 6.4: Four levels from the reconstructed pressure field produced by the annular aperture described in this experiment. Figures (a)–(d) correspond to the Radon transforms in Figure 6.3 for levels 15, 30, 45, and 60. Note the annular nature of the field.
Figure 6.5: Four level slices of the acoustic field produced by the annular aperture described in this experiment. Figures (a)–(d) correspond to levels 75, 90, 105, and 120 in Figure 6.3. Note the ring artifact in the reconstruction of level 120 due to insufficient filtering of corruption that varies only slowly from view to view. A different linear gray scale is used at each level to display the wide dynamic range of the data without distorting feature shape.
Figure 6.6: Surface plot of the reconstructed instantaneous pressure on the ultrasound pulse for level 100 (row 100 in the views shown in Figure 6.2). The pulse is moving through the plane at a slight angle.
Figure 6.7: Reconstructed views of the field of the annular aperture sliced approximately normal to transducer face. Views move through the pulse in a direction approximately parallel to the transducer emitting surface beginning with the upper left figure and progressing left to right. The annular nature of the field is very apparent here. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape.
driven with six cycles of a 70 mV sine wave at its center frequency through a 46 db Power amplifier. The diffraction plane is approximately 9.94 cm from the transducer plane. The wavelet filtering algorithm was applied to levels below 120 out of 208. Filtering was performed recursively to 3 levels (zeroing the horizontal detail image at each level). Filtered diffraction plane images are shown in Figure A.2. A total of 101 views spaced 1.8 degrees apart in angle were processed for this data set. The diffraction plane intensity views in Figure A.2 correspond to angles 0, 21.6, 43.2, 64.8, 86.4, 108, 129.6, 151.2, and 172.8 degrees. Figure A.3 shows the corresponding optical phase immediately after passing through the ultrasound pulse as computed via 40 iterations of the phase retrieval algorithm. Clearly evident are the edge waves from the straight portion of the semi-circular aperture. Each view is 34.0 mm in the horizontal and 14.3 mm in the vertical dimension. Radon transforms of several planar slices through the pulse are shown in Figure A.4. Corresponding reconstructions are in Figures 6.8 and 6.9. Figure 6.10 shows slices through the pulse which are approximately perpendicular to the emitting surface of the transducer.
Figure 6.8: Figures (a)–(d) depict tomographic reconstructions of the field produced by a semi-circular aperture for levels corresponding to rows 45, 60, 75, and 90 respectively in the views shown in Figure A.3. A different linear gray scale is used at each level to display the wide dynamic range of the data without distorting feature shape.
Figure 6.9: Figures (a)-(d) depict tomographic reconstruction of the field produced by a semi-circular aperture for levels corresponding to rows 105, 120, 135, and 150 respectively in the views shown in Figure A.3.
Figure 6.10: Reconstructed views of the field produced by six cycles at 2.25 MHz through a semi-circular aperture sliced approximately normal to the transducer face. Views move through the pulse in a direction approximately parallel to the transducer emitting surface beginning with the upper left figure and progressing left to right. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. This leads to a higher apparent level of noise in the upper left image. The peak signal in this image is low compared to the others because the data is on the periphery of the pulse.

6.3 Wide Bandwidth Double Slit: Narrow Spacing

Elements 32 and 34 of a 2.5 MHz, 64 element linear array were used to create a double slit pattern. Eight cycles of a 370 mV sine wave through a 50 db power amplifier were applied to the two elements. All other elements were grounded. The pattern was imaged approximately 12 µs into its temporal evolution. The wavelet filtering algorithm was applied to levels below 100 out of 200. Filtering was
performed recursively to 3 levels (zeroing the horizontal detail image at each level). Several filtered, background corrected views of the optical intensity approximately 7.86 cm from the transducer plane (Plane $p_1$ in Figure 2.1) are shown in Figure A.5. Each view is 34.0 mm in the horizontal and 13.7 mm in the vertical dimension. Corresponding computations of the optical phase immediately after passage through the ultrasound field are shown in Figure A.6. The interference pattern is very pronounced. Section 7 compares the reconstructed pressure obtained for this transducer to that obtained by using a hydrophone measurement. Figures A.7 shows the Radon transform of several levels of the ultrasound field. The finely detailed structure of the field is indicative of high spatial frequency patterns. Section 3.4 discusses some sampling considerations for tomographic imaging. The angular bandwidth of this particular field is higher than those discussed previously due to the short, straight line segments evident in the reconstructions shown in Figures 6.11 and 6.12. A minor artifact resulting from moderate under sampling in angle can be seen in these images as the line straight segments appear to have small amplitude extensions to the edge of the reconstruction (see Sec. 3.4 and Figure 3.10).
Figure 6.11: Figures (a)–(d) show the inversion of the Radon transforms in Figure A.7 for levels 57, 74, 91, and 108 respectively of the double slit pattern described in this experiment. Some angular under-sampling is apparent via the small amplitude extensions of straight line segments in the image (see Sec. 3.4 and Figure 3.10). A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape.
Figure 6.12: Figures (a)–(d) depict inversion of the Radon transforms in Figure A.7 for levels 125, 142, 159, and 176 respectively. Very little evidence of the ridge artifact is seen except in the very low signal image of level 176. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape.
Figure 6.13: The above are reconstructed views of the field produced by eight cycles at 2.5 MHz through two narrowly separated array elements. The images slice the field normal to the transducer face. Views move through the pulse in a direction approximately parallel to the transducer emitting surface beginning with the upper left figure and progressing left to right. The first view is very near the edge of the pulse and thus has a apparent noise level is higher than in the other images. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. This leads to a higher apparent level of noise in the upper left image. The peak signal in this image is low compared to the others because the data is on the periphery of the pulse.

6.4 Wide Bandwidth Double Slit: Wide Spacing

Elements 44 and 21 of a 2.5 MHz, 64 element linear array were used to create the field for this experiment. Both elements were driven with eight cycles of a 370 mV sine wave at the transducer
center frequency through a 50 db power amplifier. The pattern was imaged approximately 6 μs into its temporal evolution. The wavelet filtering algorithm was applied to levels below 65 out of 160. Filtering was performed recursively to 3 levels (zeroing the horizontal detail image at each level). Figure A.8 shows several diffraction plane optical intensity images taken approximately 8.68 cm after passage through the acoustic field. Each view is 34.0 mm in the horizontal and 11.0 mm in the vertical dimension. The optical phase in the plane immediately subsequent to the scattering event, computed via 40 iterations of the phase retrieval algorithm is shown for several views in Figure A.9. Evident in some of the views is an apparent artifact due perhaps to incomplete background subtraction or perhaps spatial under sampling of the field in this region. It is of note that under propagation to the transducer face and beyond (see Figure 8.6 in Sec. 8) these artifacts pass through the plane of the transducer emitting surface outside the known aperture producing the sound field. This provides a possible means of filtering the data to remove such artifacts. Note the rather detailed interference pattern at the center of the first view (upper left image) in Figure A.9. Radon transforms and their corresponding reconstructions are shown in Figures A.10, 6.14 and 6.15 respectively. Several planar slices through the field (approximately perpendicular to the transducer surface) are given in Figure 6.16.
Figure 6.14: Figures (a)–(d) show tomographic reconstructions of the field produced by a two widely separated elements of a 2.5 MHz, 64 element linear array transducer driven with eight cycles at the transducer center frequency. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. The levels are planes approximately parallel to the transducer face. Figures (a)–(d) correspond to rows 17, 34, 51, and 68 of the views in Figure A.9 respectively.
Figure 6.15: The above images show tomographic reconstructions of the field produced by two widely separated elements of a 2.5 MHz, 64 element linear array transducer driven with eight cycles at the transducer center frequency. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. The levels are planes approximately parallel to the transducer face. Figures (a)–(d) correspond to rows 85, 102, 119, and 136 of the views in Figure A.9 respectively.
6.5 Tissue Scattering

In this section we present data indicating the utility of this imaging technique for studying the interaction of ultrasound with materials such as biological tissue. Two imaging experiments are performed. In the first (described in Figures 6.17, 6.19, A.11, A.13, A.15) an unfocused, 2.25 MHz, 12.5 mm diameter Panametrics ultrasound transducer is driven with six cycles of a 70 mV sine wave through a 46 db power amplifier. The pulse was imaged approximately 19 μs into its temporal evolution. A total

Figure 6.16: Planar slices of the ultrasound pulse with slice normals approximately perpendicular to the transducer surface normal. Views move through the pulse in a direction approximately parallel to the transducer emitting surface beginning with the upper left figure and progressing left to right. This leads to a higher apparent level of noise in the upper left image. The peak signal in this image is low compared to the others because the data is on the periphery of the pulse.
of 101 views spaced 1.8 degrees apart in angle were taken. The diffraction plane location (Plane $p_2$ in Figure 2.1) was approximately 9.84 cm subsequent to passage through the ultrasound pulse (Plane $p_1$ in Figure 2.1). The wavelet filtering algorithm was applied to levels below 105 out of 200. Filtering was performed recursively to 3 levels (zeroing the horizontal detail image at each level) in order to reduce ridge artifacts. A physical view field of 34.0 mm in the horizontal and 13.7 mm in the vertical dimension was used. In the second experiment (described in Figures 6.18, 6.20, A.12, A.14, A.16) the same diffraction plane distance, view field, angular positions, and drive wave form were used. However, an approximately 150 mm $\times$ 150 mm $\times$ 150 mm piece of beef with obvious marbling was affixed to the transducer surface using commercial ultrasound impedance matching gel and a latex condom. The beef cube was situated so as to cover the entire surface of the transducer and all visible air pockets where eliminated.

The optical intensities in Plane $p_2$ of Figures 2.1 and 3.2 for the aberrated and unaberrated cases are shown in Figures A.12 and A.11 respectively. Applying 40 iterations of the phase retrieval algorithm to these images yields the optical phase in the plane immediately after passage through the ultrasound pulse. These data are shown in Figures A.14 and A.13 for the aberrated and unaberrated views respectively. Clearly evident are the wavefront distortions produced by the presence of the tissue. Also a bulk longitudinal wave speed difference between the beef and water is clearly shown.

The Radon transforms for several planar slices of the field approximately parallel to the transducer surface are shown in Figures A.15 and A.16 for the unaberrated and aberrated cases respectively. These are taken from roughly similar locations in the unaberrated and aberrated pulses. The corresponding reconstructions are shown in Figures 6.17, and 6.19 for the unaberrated aperture and in Figures 6.18, and 6.20 for the aberrated case.
Figure 6.17: Tomographic reconstruction of the unaberrated ultrasound pulse. Figures (a)–(d) correspond respectively to rows 17, 34, 51, and 68 in the views shown in Figure A.13. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. Evidence of incomplete background removal is indicated in each of the image by the concentric right pattern located at the image center. Figure 6.18 shows reconstructions of similar regions of the ultrasound field in the aberrated case.
Figure 6.18: The above images show tomographic reconstructions of the aberrated ultrasound pulse. Figures (a)–(d) correspond respectively to rows 37, 54, 71, and 88 in the views shown in Figure A.14. Figure 6.17 shows reconstructions of similar regions of the ultrasound field in the unaberrated case. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. The 20 slice difference in the displayed reconstructions is due to a difference in the bulk wave speeds of beef and water.
Figure 6.19: The above images show tomographic reconstructions of the unaberrated ultrasound pulse. Figures (a)–(d) correspond respectively to rows 85, 102, 119, and 136 in the views shown in Figure A.13. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. Figure 6.20 shows reconstructions of similar regions of the ultrasound field in the aberrated case.
Figure 6.20: The above images are tomographic reconstructions of the unaberrated ultrasound pulse. Figures (a)–(d) correspond respectively to rows 105, 122, 139, and 156 in the views shown in Figure A.14. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. Figure 6.19 shows reconstructions of similar regions of the ultrasound field in the unaberrated case.
In this section a comparison between data obtained with an NTR Systems, Inc. 500 μm diameter hydrophone and the optical imaging technique discussed in this paper is presented. The narrowly separated double slit pattern obtained by exciting two nearby elements of a linear array discussed in Secs. 6.3 and 8.2 is used. The relevant parameters of that experiment are as follows. Elements 32 and 34 of a 2.5 MHz, 64 element linear array were used to create a double slit pattern. Eight cycles of a 370 mV sine wave through a 50 db power amplifier were applied to the two elements. All other elements were grounded. The pattern was imaged approximately 12 μs into its temporal evolution. The wavelet filtering algorithm was applied to levels below 100 out of 200. Filtering was performed recursively to 3 levels (zeroing the horizontal detail image at each level). The optical intensity was imaged approximately 7.86 cm from the transducer plane. Each view is 34.0 mm in the horizontal and 13.7 mm in the vertical dimension.

The same drive signals were used to generate the pulse for the hydrophone experiment. Due to the difficulty of obtaining accurate three dimensional hydrophone scans at the resolution needed to compare with the optical imaging technique, a two-dimensional pressure time series was measured in a plane and the theory described in Sec. 2.8 was used to generate a three-dimensional spatial distribution for comparison. During the measurement process the hydrophone was stepped 199 times (after measurement at the initial position) in increments of 0.2 mm each in both the horizontal dimensions. The 200 samples of the time series in the measurement plane (which was approximately parallel to the emitting surface of the transducer) were taken at a sampling rate of 20 MHz.

In order to verify the viability of the method of described in Sec. 2.8, time series data was taken for one, two, and three element separations of the two driven array elements. The time series was then propagated back to the emitting surface of the transducer to obtain an estimate of the drive signal on the elements. Use of a source-free model is not required here (as discussed in Sec. 2.8). However, as is typically the case with inverse holographic or back propagation techniques, high frequency noise in the measurement plane grows exponentially and requires some kind of regularization or truncated inverse solution. In these calculations evanescent field contributions are not included. Figures 7.1, 7.2, and 7.3 show the calculated time series needed in the plane of the transducer emitting surface to produce the time series observed in the measurement plane.

The optical data for the narrowly separated double slit experiment using array elements 32 and 34 were taken at an approximate temporal delay of 12 μs. The time series data taken for the hydrophone were taken at approximately 5 μs. Thus good registration is expected between the optical and hydrophone measurements if the time series data are used to compute the spatial pressure distribution at an additional delay of 7 μs. It was visually determined that the best temporal registration occurred for an additional 7.5 μs delay on the hydrophone data. The hydrophone data were propagated to 200 consecutive data planes roughly covering the region of space where the ultrasound pulse would be
for a delay of 7.5 µs. The time points corresponding to this delay were extracted from each data plane to form the three-dimensional spatial pressure distribution. The transverse dimensions (those of the hydrophone measurement plane) were then scaled and interpolated according to the known increments used for the hydrophone measurement process. The direction approximately parallel to the transducer surface normal was scaled to match zero crossings and peaks (this is the direction involving the optical system scale factor). Two slices from approximately the same location in the hydrophone and optical data volumes were selected by hand. Each is normalized so that the largest magnitude in each image is unity. Figures 7.4(a) and (b) show the hydrophone and laser data slices respectively. Line plot comparisons between the two slices are shown in Figures 7.5 and 7.6. Excellent agreement is observed for all but the last subfigure in each series. These data are in the region of the field where hydrophone measurement becomes difficult due to the directivity pattern of the hydrophone aperture and the incidence angle of the impinging waves.
Figure 7.1: Figures (a)–(d) show back propagated hydrophone time series data (approximately the time series in the transducer surface plane) using elements 30 and 34 of the 2.5 MHz linear array transducer. A 500 μm diameter needle hydrophone was used to acquire the data in the measurement plane. The location of the transducer plane was deduced visually by computing the time series for several possible transducer surface distances near 1 cm (the approximate distance at which the time series was measured). (a) A slice of the pressure time series applied to the plane of the emitting surface of the transducer. The vertical dimension of the image is a time coordinate and the horizontal a spatial position coordinate. (b) A view with spatial coordinate orthogonal to that in (a). (c) A line plot of the data under the horizontal line in (a). (d) A line plot of the data under the vertical line in (b).
CHAPTER 7. HYDROPHONE COMPARISON

Figure 7.2: Figures (a)–(d) show back propagated hydrophone time series data (approximately the time series in the transducer surface plane) using elements 31 and 34 of the 2.5 MHz linear array transducer. A 500 μm diameter needle hydrophone was used to acquire the data in the measurement plane. The location of the transducer plane was deduced visually by computing the time series for several possible transducer surface distances near 1 cm (the approximate distance at which the time series was measured). (a) A slice of the pressure time series applied to the plane of the emitting surface of the transducer. The vertical dimension of the image is a time coordinate and the horizontal a spatial position coordinate. (b) A view with spatial coordinate orthogonal to that in (a). (c) A line plot of the data under the horizontal line in (a). (d) A line plot of the data under the vertical line in (b).
Figure 7.3: Figures (a)–(d) show back propagated hydrophone time series data (approximately the time series in the transducer surface plane) using elements 32 and 34 of the 2.5 MHz linear array transducer. A 500 μm diameter needle hydrophone was used to acquire the data in the measurement plane. The location of the transducer plane was deduced visually by computing the time series for several possible transducer surface distances near 1 cm (the approximate distance at which the time series was measured). (a) A slice of the pressure time series applied to the plane of the emitting surface of the transducer. The vertical dimension of the image is a time coordinate and the horizontal a spatial position coordinate. (b) A view with spatial coordinate orthogonal to that in (a). (c) A line plot of the data under the horizontal line in (a). (d) A line plot of the data under the vertical line in (b).
Figure 7.4: Description of lines used in comparison of hydrophone and optical measurements. (a) A slice (approximately normal to the transducer face) through the three-dimensional spatial distribution of pressure. The field was computed from the time series measured in a plane with a 500 μm diameter hydrophone. Indicated on the plot are the data used in the line plot comparisons of Figure 7.5(a)-(d). (b) A slice of the reconstructed laser imaged ultrasound field (approximately the same location as the hydrophone data in Figure 7.4(a)). Indicated on the plot are the data used in the line plot comparisons in Figure 7.6(a)-(d).
Figure 7.5: Comparison of three-dimensional data volumes computed from hydrophone and laser measurements. Locations in the hydrophone and laser data sets are in Figure 7.4(a). (a)–(c) Line plots indicating excellent agreement between hydrophone and laser measurements. (d) A line plot in the back region of the pulse. The hydrophone has difficulty here due to the incidence angle of impinging acoustic wavefronts.
Figure 7.6: Comparison of the three-dimensional data volumes computed from hydrophone and laser measurements. Locations in the hydrophone and laser data sets are in Figure 7.4(b). (a)–(c) Line plots indicating excellent agreement between hydrophone and laser measurements. (d) A line plot in the back region of the pulse. The hydrophone has difficulty here due to the incidence angle of impinging acoustic wavefronts.
Propagation sequences and particle velocity and Poynting vector fields for tomographic views (see Appendix B) of several wide bandwidth fields are presented in this chapter. Section 8.1 discusses computations for a field created by a wide bandwidth drive signal on elements 27, 31, 34, 36, and 38, together with a 180 degree phase shifted version of the same signal on elements 28, 30, 32, 33, 35, and 37 of a 64 element, 2.5 MHz linear array transducer. The same transducer is used to create a pulsed double slit pattern using elements 32 and 34 in Sec. 8.2 and elements 44 and 21 in Sec. 8.3. Section 8.4 demonstrates propagation and particle velocity and Poynting vector fields for a semi-circular aperture driven with an unfocused 12.5 mm diameter, 2.25 MHz transducer.

8.1 Complicated Field Pattern

Figures 8.2 and 8.1 depict the time series propagation for a “checkered” field pattern. The original two-dimensional projection of the ultrasound pulse (a single measured view) is shown in Figure 8.1 for reference purposes. Each view is 34.0 mm in the horizontal and 12.4 mm in the vertical dimension. A diffraction plane distance (the separation between Plane \( p_1 \) and Plane \( p_2 \) in Figure 2.1) of 8.68 cm was used and the wavelet filtering algorithm was applied to levels below 65 out of 180 of the entire data set before selecting the first view for demonstrating spatial propagation. Filtering was performed recursively to 3 levels (zeroing the horizontal detail image at each level). The phase retrieval algorithm was run for 40 iterations to compute the optical phase immediately after passage through the sound field. The pulse was created using a 64 element, 2.5 MHz, linear array. Elements 27 through 38 were used (all other elements were grounded). A basic signal consisting of eight cycles of a 370 mV sine wave at the center frequency of the transducer through a 50 db power amplifier was used. A phase inverter created a 180 degree phase shifted version of this signal. Elements 33 through 38 were driven, alternating drive signals between the basic and 180 degree phase shifted versions every other element. This same pattern was used for elements 27 through 32. The pulse was imaged approximately 9.6 \( \mu \)s into its temporal evolution. After completion of the experiment it was discovered that element 29 had a faulty wire connection and was not being driven at all. This is clear from the propagation sequence shown in Figure 8.2. Also, the two central elements are clearly in phase. The approximate location of the transducer face is depicted on the images via the two horizontal lines. Note that the space above these lines is non-physical. That is, the signal in this space may be regarded as the pulse with source at infinity that would produce the measured pulse in front of the transducer.
Figure 8.1: A measured projection of the “Checkered” field used to create the time sequence depicted in Figure 8.2.
Figure 8.2: Propagation sequence computed from the pulse shown in Figure 8.1. The pulse was created using 12 central elements of a 64 element, 2.5 MHz linear array with three drive signals: eight cycles at the transducer center frequency, the 180 degrees phase delayed version of this signal and one element not driven at all. The original pulse was imaged 9.6 $\mu$s into its temporal evolution.

8.2 Wide Bandwidth Double Slit: Narrow Spacing

A time sequence for the narrowly separated, wide bandwidth, double slit pattern discussed in Sec. 6.3 is shown in Figure 8.4. For the benefit of the reader, the experimental parameters are described here again. Elements 32 and 34 of a 2.5 MHz, 64 element linear array were used to create a double slit pattern. Eight cycles of a 370 mV sine wave through a 50 db power amplifier were applied to the two elements. All other elements were grounded. The pattern was imaged approximately 12 $\mu$s into its temporal evolution. The wavelet filtering algorithm was applied to levels below 100 out of 200.
Filtering was performed recursively to 3 levels (zeroing the horizontal detail image at each level). The optical intensity was imaged approximately 7.86 cm from the transducer plane. Each view is 34.0 mm in the horizontal and 13.7 mm in the vertical dimension. Figure 8.3 shows the optical phase immediately after passage through the ultrasound pulse as computed via 40 iterations of the phase retrieval algorithm. This view was used to compute the time sequence depicted in Figure 8.4. The approximate location of the transducer plane is indicated by the two horizontal lines in each frame. An indication of accurate measurement is the tightly confined aperture defined by the bulk of the acoustic energy as it passes through the plane of the transducer. Note, however, that there is a small amount of energy that passes through the transducer plane clearly outside the emitting region of the transducer indicating the possibility of improving the signal further by model-based filtering of this time sequence.

**Figure 8.3:** A measured projection of the narrowly separated double slit field used to create the time sequence depicted in Figure 8.4.
**Figure 8.4:** Time sequence for a single projection of the narrowly separated double slit pattern discussed in Sec. 6.3 computed from the optical phase view in Figure 8.3. Each view is 34.0 mm in the horizontal and 13.7 mm in the vertical dimension. The location of the transducer surface is indicated in each frame by the two horizontal lines.

### 8.3 Wide Bandwidth Double Slit: Wide Spacing

A time sequence for a single view of the widely separated double slit pattern discussed in Sec. 6.4 is shown in Figure 8.6. The optical phase image used to compute the projection of the pulse at different times is shown in Figure 8.5. For the benefit of the reader the parameters describing the experiment are repeated here. Elements 44 and 21 of a 2.5 MHz, 64 element linear array were used to create the field for this experiment. Both elements were driven with eight cycles of a 370 mV sine wave at the transducer center frequency through a 50 db power amplifier. The pattern was imaged approximately
into its temporal evolution. The wavelet filtering algorithm was applied to levels below 65 out of 160. Filtering was performed recursively to 3 levels (zeroing the horizontal detail image at each level). Optical intensity images were taken approximately 8.68 cm after passage through the acoustic field. Each view is 34.0 mm in the horizontal and 11.0 mm in the vertical dimension. The optical phase in the plane immediately subsequent to the scattering event was computed via 40 iterations of the phase retrieval algorithm (see Figures 8.5 and A.9). Again, the acoustic energy passes through the transducer plane (indicated by the two horizontal lines in each frame of Figure 8.6) in two tightly confined regions. Also, as in the narrowly separated double slit case, we see the possibility of reducing artifacts in the signal by filtering out portions of the signal passing through the plane of the transducer emitting surface in regions clearly outside the transducer aperture.

![Figure 8.5](image)

**Figure 8.5:** The optical phase view (projection of the acoustic field) used for computing the time sequence in Figure 8.6. Each view is 34.0 mm in the horizontal and 11.0 mm in the vertical dimension. Other projections of the pulse are shown in Figure A.9. The rectangles indicate regions where particle velocity and Poynting vector fields are computed for Figures 8.7, 8.8, 8.9, 8.10, 8.11, and 8.12.
Figure 8.6: A time sequence computed from the optical phase image in Figure 8.5. Elements 44 and 21 of a 2.5 MHz, 64 element linear array were used to create the field for this experiment. Both elements were driven with eight cycles at the center frequency of the transducer. The pattern was imaged approximately $6 \mu s$ into its temporal evolution. Each view is 34.0 mm in the horizontal and 11.0 mm in the vertical dimension.
Figure 8.7: A subregion of the instantaneous Poynting vector field created by an aperture consisting of two rectangular slits overlayed on the original pressure field. The subregion used is indicated by boxed region 1 in Figure 8.5.

Figure 8.8: A subregion of the instantaneous velocity field created by an aperture consisting of two rectangular slits overlayed on the original pressure field. The subregion used is indicated by boxed region 1 in Figure 8.5.
**Figure 8.9:** A subregion of the instantaneous Poynting vector field created by an aperture consisting of two rectangular slits overlayed on the original pressure field. The subregion used is indicated by boxed region 2 in Figure 8.5.

**Figure 8.10:** A subregion of the instantaneous velocity field created by an aperture consisting of two rectangular slits overlayed on the original pressure field. The subregion used is indicated by boxed region 2 in Figure 8.5.
8.4 Semi-Circular Aperture

Propagated images of the semi-circular aperture discussed in Sec. 6.2. Again, the relevant parameters for that experiment are as follows. Several layers of absorbing tape were used to cover half the aperture of an unfocused, 2.25 MHz, 12.5 mm diameter, Panametrics transducer thus producing a semi-circular aperture. The transducer was driven with six cycles of a 70 mV sine wave at its center frequency through a 46 db Power amplifier. The wavelet filtering algorithm was applied to levels below 120 out of 208. Filtering was performed recursively to 3 levels (zeroing the horizontal detail image at each level). Figure 8.13 shows the corresponding optical phase immediately after passing through the ultrasound pulse as computed via 40 iterations of the phase retrieval algorithm for a single view (for other views see Figure A.3). The view field is 34.0 mm in the horizontal and 14.3 mm in the vertical
dimension. In the view used for the time sequence the straight portion of the aperture is oriented normal to the plane of the page. Thus we expect to see an end on view of a line-like source present in the images. This is clearly evident in Figure 8.14, particularly when considering the time points prior to passage through the transducer plane.

Figure 8.13: The above image shows the measured optical phase immediately after passage through the ultrasound pulse. It is used for computing the time sequence shown in Figure 8.14. The straight portion of the aperture is oriented normal to the plane of the page. The view field is 34.0 mm in the horizontal and 14.3 mm in the vertical dimension. See Figure A.3 for other views. The rectangle indicates the region used in the Poynting vector and particle velocity field computations shown in Figures 8.15 and 8.16 respectively.
Figure 8.14: A time sequence computed from the projection of the ultrasound pulse shown in Figure 8.13. The straight portion of the aperture is oriented normal to the plane of the page. Note the presence of a line-like source in the field due to this edge. The view field is 34.0 mm in the horizontal and 14.3 mm in the vertical dimension.
Figure 8.15: A subregion of the instantaneous Poynting vector field created by a semi-circular aperture overlayed on the original pressure field. The subregion used is indicated in Figure 8.14.

Figure 8.16: A subregion of the instantaneous velocity field created by a semi-circular aperture overlayed on the original pressure field. The subregion used is indicated in Figure 8.14.
SUMMARY

This chapter briefly reviews the material presented in Chapters 2 through 8, discusses the assertions of the thesis hypothesis (see page 1), and states the primary contributions of this thesis to the areas of optical measurement of acoustic fields, optical tomography, and multi-dimensional signal processing.

This thesis describes an optical method for detailed three-dimensional measurement of instantaneous pressure in a wide bandwidth ultrasound pulse. The method records the forward-scattered optical intensity from the interaction of a collimated laser pulse with an ultrasound field via a two-lens imaging system (see Sec. 2.1). The optical intensity is recorded at several angles as the ultrasound transducer is rotated through 180 degrees about an axis normal to the propagation direction of the incident optical pulse. The phase of the optical field immediately after passage through the ultrasound pulse is computed from this intensity via the Method of Generalized Projections (MGP) (see Sec. 3.3) under a scalar Fresnel model for optical wave propagation (see Sec. 2.2). A simple delay model is described in Sec. 2.4 for the interaction of the optical wave with the ultrasound pulse. This allows the reconstruction of the instantaneous three-dimensional refractive index distribution from the optical phase via the Fourier Slice Reconstruction (FSR) algorithm (see Secs. 2.1 and 3.4). A simple linear model for the relationship between local pressure and local refractive index perturbation is described in Section 2.3. It is shown in Sec. 2.5 that the spatial sampling rate for the intensity images must be at least $4 \max(B_t)/c$, where $c$ is the wave speed in the medium and $\max(B_t)$ is the highest temporal frequency passed by the transducer. A rigorous model for acoustic wave propagation is reviewed in Sec. 2.6 precisely defining pressure as the average of the diagonal elements of the internal stress tensor.

In section 2.7.1 it is shown that a single measurement of instantaneous pressure together with the assumption that the wide bandwidth angular plane wave spectrum of the ultrasound pulse consists only of forward-propagating waves (wave vector components normal to the transducer surface are greater than zero) provides a means of propagating the measurement forward or backward in time under a source-free model. In Sec. 2.7.2 a new sampling theorem is derived stating that two complete measurements of the three-dimensional pressure field separated in time by $\Delta t$ allow release of the forward-propagating assumption for every acoustic wave number $k$ satisfying $k \neq n\pi/(c\Delta t)$, where $c$ is the acoustic wave speed in the medium and $n$ an integer greater than zero. Thus, the temporal evolution of very general ultrasound fields may be computed from a minimal number of measurements. Section 2.8 describes how the pressure time series over a plane may be propagated to any other parallel plane. This is very useful for making hydrophone measurements of a field. It is demonstrated in Chapter 7 and used to compute the three-dimensional pressure distribution in an ultrasound pulse for comparison with an optical measurement of the same field. Sections 2.9 and 2.10 discuss the computation of Poynting vector and particle velocity fields respectively.
Chapter 3 discusses signal processing aspects of the measurement. Section 3.1 explains the relationship between the physical optical intensity and the digitized signal. Section 3.2 describes a two-step method for reducing image background corruption in the optical intensity recording resulting from dust particles in the beam path, optical element defects, optical beam nonuniformity, multiple reflections in the optical imaging path and similar deterministic sources of error that vary slowly with tomographic view. As mentioned above, pressure is reconstructed from optical phase. Optical phase is computed from optical intensity as the solution of a fixed point equation via the Method of Generalized Projections (MGP) (see Sec. 3.3.2). Only a very small class of signals satisfies the fixed point equation making the computed phase function unique up to a constant, unit magnitude, complex coefficient (see Sec. 3.3.1). Section 3.4 discusses the Fourier Slice Reconstruction (FSR) algorithm, the spatial sampling rate for the optical intensity images, and the angular sampling rate necessary for accurate reconstruction of pressure in the ultrasound pulse.

Chapter 4 simulates the entire experiment. Generation and propagation of the ultrasound field are computed via a Fresnel model for acoustic wave propagation. A scalar Fresnel model for optical wave propagation (see Sec. 2.2.5) and a simple delay model for optical scattering (see Sec. 2.4) are used to compute the forward-scattered optical field at each of several angles as the ultrasound pulse is rotated about an axis normal to the propagation direction of the incident optical wave. Gaussian white noise is added to the simulated intensity images to yield a signal peak to noise variance ratio of $8.6 \times 10^{-6}$ (estimated from experimental data). The MGP (Sec. 3.3.2) is used to compute the phase of the optical wave immediately after passage through the acoustic field and the Fourier Slice Reconstruction (FSR) algorithm (see Sec. 3.4) used to reconstruct the numerical ultrasound pulse phantom. The reconstructed pressure field is compared with the original model and seen to agree very well.

Calibration of the physical system and investigation of the linearity of the instrumentation used to record optical intensity is covered in Sec. 5.1. Linearity of the measurement process itself is examined in Secs. 5.2.1 and 5.2.2 by explicit experimental testing of the superposition and scaling properties satisfied by linear systems. Sections 5.3.1 and 5.3.2 demonstrate the phase retrieval and reconstruction process. The latter examines the effectiveness of a wavelet-based filter (see Sec. 3.2.2) in removing a particular type of background corruption.

Chapter 6 gives complete reconstructions of the instantaneous three-dimensional pressure distributions in several different fields. Sections 6.1 and 6.2 respectively cover fields from annular and a semi-circular apertures driven with an unfocused 2.25 MHz transducer. Sections 6.3 and 6.4 respectively discuss reconstruction of a wide bandwidth field produced with elements 32 and 34 of a 64 element, 2.5 MHz linear array transducer, and elements 44 and 21 of the same. Full measurements of a wide bandwidth field from a 12.5 mm diameter, 2.25 MHz, unfocused transducer with and without a cube of beef (approximately 150 mm × 150 mm × 150 mm in size) in front of the transducer are presented and discussed in Sec. 6.5.

Section 7 presents hydrophone measurements of three separate wide bandwidth fields produced using elements 30 and 34, elements 31 and 34, and elements 32 and 34 of a 64 element, 2.5 MHz linear array transducer. A 500 μm diameter needle hydrophone was used to measure the pressure time series in a plane approximately parallel to the surface of a the transducer and the theory in Sec. 2.8 is used to propagate the three time series to the surface of the transducer. The shape and dimensions of the fields in the plane of the transducer emitting surface match the drive signals and known source parameters well. The same method is used to compute the instantaneous three-dimensional pressure distribution for the experiment using elements 32 and 34 of the 64 element, 2.5 MHz linear array transducer. This field was measured at approximately the same delay via the optical method presented in this thesis. Hydrophone and optical measurements are registered by hand and pressure line plot comparisons agree very well. The greatest disparity occurs in regions of the field where the directivity pattern of the hydrophone complicates accurate measurement (those containing wavefronts impinging on the measurement surface of the hydrophone with significant incidence angles).

Chapter 8 contains propagation sequences and particle velocity and Poynting vector fields for tomographic views (see Appendix B) of wide bandwidth fields for the following sources: a semi-circular aperture driven with an unfocused 12.5 mm diameter, 2.25 MHz transducer, elements 32 and 34 of a 64 element, 2.5 MHz linear array transducer, elements 44 and 21 of the aforementioned linear
array, and elements 27, 31, 34, 36, and 38, together with a 180 degree phase shifted version of the same signal on elements 28, 30, 32, 33, 35, and 37 of the aforementioned linear array.

Appendix B shows that the Radon transform of the solution to an $N$-dimensional, linear, shift-invariant operator equation solves an $(N-1)$-dimensional equation. This allows parallel computation of the solution to such problems. It is used in this thesis to compute the numerical ultrasound field phantom used in the simulation discussed in Sec. 4. Appendix C derives a wave optics model for the optical measurement system. Appendix D defines the operator notation used in Appendix C.

**Hypothesis summary.** The hypothesis (see page 1) that detailed, accurate, three-dimensional, optical measurements of an instantaneous, refractive index perturbation in an optically transparent medium may be obtained from measurements of scattered optical intensity alone is clearly supported by the results given in Chapter 6 and Appendix A. The Method of Generalized Projections (MGP) provides an effective means of incorporating these measurements into an iterative algorithm for computing the optical phase as the solution of a fixed point equation (see Sec 3.3.2 and Eq. 3.46). Section 3.3.1 shows that the complex optical field amplitude computed in this manner, is unique up to a constant unit magnitude complex coefficient. The distribution of refractive index induced by several ultrasound pulses is computed via the Fourier Slice Reconstruction (FSR) algorithm from optical phase data under the assumption of weak optical scattering in Chapter 6. A comparison with hydrophone measurements given in Chapter 7 supports the accuracy of these data. Chapter 8 demonstrates computation of Poynting vector, and particle velocity fields from experimental measurements. Temporal sequences for pressure distributions in several different pulses are shown as well. Section 6.5 compares optical measurements of an ultrasound pulse scattered by a piece of biological tissue (beef) to an non-scattered pulse created under otherwise identical experimental conditions.

**Contributions.** The contributions of this thesis to the areas of optical measurement of acoustic fields, optical tomography, and the image processing subject of phase recovery are as follows:

- A new method for fully three-dimensional, non-perturbative, high-resolution optical measurement of instantaneous pressure in wide bandwidth ultrasound fields of arbitrary structure is developed (see Chapters 2 and 3). Several different fields with complex structures are measured optically (see Chapter 6) and are found to match hydrophone measurements very well (see Chapter 7).

- A three-dimensional measurement of tissue-induced phase aberration in a wide bandwidth ultrasound pulse is obtained via the optical method presented in this thesis. Comparison of this measurement with an unaberrated pulse under otherwise identical experimental conditions is presented (see Sec. 6.5), demonstrating a means of making detailed studies of tissue-beam interaction.

- Computation of the full four-dimensional space-time dependence of an acoustic field from a single three-dimensional measurement under a forward-propagating assumption on its wide bandwidth angular plane wave spectrum is reviewed (see Sec. 2.7.1). Computations of particle velocity and Poynting vector fields are also explained (see Secs. 2.9 and 2.10 respectively). The theory is applied to several experimental data sets. Acoustic energy passing through the plane of the transducer emitting surface outside the known aperture indicates the possibility of improving the accuracy of this new optical method through application of a model-based filter (see Chapter 8).

- A new theorem is derived, stating that two complete measurements of the three-dimensional pressure field, separated in time by $\Delta t$, allow release of the forward-propagating assumption for every acoustic wavenumber $k$, satisfying $k \neq n\pi/(c\Delta t)$, where $c$ is the acoustic wave speed in the medium and $n$ an integer greater than zero. Thus, the temporal evolution of very general ultrasound fields may be computed from a minimal number of measurements (see Sec. 2.7.2).

- A result in the literature [1] on the uniqueness of a real, finite-support, multi-dimensional signal given the magnitude of its Fourier transform is extended to cover complex signals. The extended
theorem states that a complex signal is determined uniquely up to a shift, a unit magnitude complex coefficient, and conjugate reflection through the origin by the magnitude of its Fourier transform, if its $z$-transform possesses at most a single non-conjugate symmetric factor (see Sec. 3.3.1).

- Application of the Method of Generalized Projections (MGP) to compute optical phase for use in tomographic reconstruction of the ultrasound field requires understanding of the uniqueness of optical phase given intensity in the measurement experiment (see Figures 2.1, 4.1, and A.1). In Sec. 3.3.1 a complex, finite-support, multi-dimensional signal is shown to be determined uniquely up to a unit magnitude complex coefficient, if the $z$-transform of the product of the signal with a multi-dimensional chirp possesses at most a single non-conjugate symmetric factor.

- A simple wavelet-based filtering method for handling background corruption due to non-ideal characteristics of the optical beam source, dust particles, optical element flaws, and optical multipath is developed and found to work well (see Secs. 3.2 and 3.2.2).
This appendix contains additional images for the experiments reported in Chapter 6. For the convenience of the reader a diagram of the experiment is repeated below in Figure A.1. In each of the data sets optical intensity is measured in Plane $p_2$ and the optical phase in Plane $p_1$ is computed via the Method of Generalized Projections (MGP) (see Sec. 3.3.2).

\begin{center}
\includegraphics[width=\textwidth]{figure_a1.png}
\end{center}

\textbf{Figure A.1:} A description of the basic experiment. The ultrasound transducer and laser are fired with timing that allows the collision of optical and acoustic pulses to take place near the optical axis of the imaging system. The lenses $\ell_2$ and $\ell_3$ provide a means of imaging arbitrary planes within the imaging volume onto the CCD array. Plane $p_1$ is the plane immediately subsequent to passage of the optical pulse through the acoustic field. Plane $p_2$ is an arbitrary diffraction plane. The dimensions of the water tank used in this experiment are approximately 18 cm $\times$ 18 cm $\times$ 32 cm.
Figure A.2: The D-Shaped aperture diffraction plane views for angles 0, 21.6, 43.2, 64.8, 86.4, 108, 129.6, 151.2, and 172.8 degrees beginning with the upper left figure and proceeding left to right. Each view is approximately 34.0 mm in the horizontal and 14.3 mm in the vertical dimension. Compare Figure A.3.
Figure A.3: Several images of the optical phase immediately after passing through the ultrasound pulse. Forty iterations of the phase retrieval algorithm were used. Each view is 34.0 mm in the horizontal and 14.3 mm in the vertical dimension and corresponds to the similarly located diffraction plane view shown in Figure A.2.
Figure A.4: Several Radon transforms for planar slices of the ultrasound pulse from the D-shaped aperture described in this experiment. The normal for each level is parallel to the tomographic rotational axis and approximately parallel to the transducer surface normal. These levels correspond to rows 45, 60, 75, 90, 105, 120, 135, 150 in the views shown in Figure A.3 beginning with the upper left image and progressing left to right. A different linear gray scale is used at each level to display the wide dynamic range of the data without distorting feature shape. Reconstructions proportional to the instantaneous pressure in the pulse at these levels are shown in Figures 6.8 and 6.9.
Figure A.5: Diffraction plane intensity in an eight cycle double slit experiment for angles 0, 21.6, 43.2, 64.8, 86.4, 108, 129.6, 151.2, and 172.8 degrees beginning with the upper left figure and proceeding left to right. Each view is 34.0 mm in the horizontal and 13.7 mm in the vertical dimension. The diffraction plane is located approximately 13.7 cm from the plane over which we desire to know the phase. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape.
Figure A.6: Optical phase immediately after passage through the acoustic field as computed via 40 iterations of the phase retrieval algorithm. Angles 0, 21.6, 43.2, 64.8, 86.4, 108, 129.6, 151.2, and 172.8 degrees beginning with the upper left figure and proceeding left to right (Compare Figure A.5). Each view is 34.0 mm in the horizontal and 13.7 mm in the vertical dimension. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape.
Figure A.7: Radon transforms of several planar slices through the ultrasound field. Each level has a normal approximately parallel to the plane containing the two emitting elements of the transducer. Levels corresponding to rows 57, 74, 91, 108, 125, 142, 159, and 176 in the views shown in Figure A.6 are shown beginning with the upper left and proceeding across the rows. Corresponding reconstructions are shown in Figures 6.11 and 6.12. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape.
Figure A.8: The above images show diffraction plane optical intensity for two widely separated elements of a linear array transducer at angles 0, 21.6, 43.2, 64.8, 86.4, 108, 129.6, 151.2, and 172.8 degrees beginning with the upper left figure and proceeding left to right. Each view is 34.0 mm in the horizontal and 11.0 mm in the vertical dimension. The plane in which the intensity is pictured is approximately 8.68 cm subsequent to passage through the ultrasound field. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. Compare Figure A.9.
Figure A.9: The above images show optical phase immediately after passage through the ultrasound field from two widely separated elements of a linear array transducer at tomographic view angles 0, 21.6, 43.2, 64.8, 86.4, 108, 129.6, 151.2, and 172.8 degrees beginning with the upper left figure and proceeding left to right. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. Compare Figure A.8.
Figure A.10: Radon transforms of several planar slices through the ultrasound field. Each level has a normal approximately parallel to the plane containing the two emitting elements of the transducer. Levels corresponding to rows 17, 34, 51, 68, 85, 102, 119, and 136 in the views shown in Figure A.9 are shown beginning with the upper left and proceeding across the rows. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. Corresponding reconstructions are shown in Figures 6.14 and 6.15.
Figure A.11: The above images show optical intensity approximately 9.84 cm after passage through the ultrasound field for an unaberrated, unfocused, 2.25 MHz, 12.5 mm diameter transducer. The pulse was created with six cycles at 2.25 MHz and imaged approximately 19 μs into its temporal evolution. Views are shown for angles 0, 21.6, 43.2, 64.8, 86.4, 108, 129.6, 151.2, and 172.8 degrees beginning with the upper left image and proceeding left to right. Figure A.12 shows the corresponding views for the aberrated aperture experiment. Note the apparent difference in bulk wave speed. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape.
Figure A.12: The above images show optical intensity in a plane 9.84 cm after passage through the ultrasound field for an aberrated, unfocused, 2.25 MHz, 12.5 mm diameter transducer. The pulse was created with six cycles at 2.25 MHz and imaged approximately 19 µs into its temporal evolution. Views are shown for angles 0, 21.6, 43.2, 64.8, 86.4, 108, 129.6, 151.2, and 172.8 degrees beginning with the upper left figure and proceeding left to right. Figure A.11 shows the corresponding views for the experiment with no aberrator affixed to the emitting surface of the transducer. Note the apparent difference in bulk wave speed. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape.
Figure A.13: Optical phase in the plane immediately subsequent to passage through the ultrasound field (no acoustic phase aberration) as computed via 40 iterations of the phase retrieval algorithm. Views are shown for angles 0, 21.6, 43.2, 64.8, 86.4, 108, 129.6, 151.2, and 172.8 degrees corresponding to the diffraction plane optical intensity shown in Figure A.11. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. Figure A.14 shows the corresponding views for the experiment with the aberrator affixed to the emitting surface of the transducer.
**Figure A.14:** Optical phase in the plane immediately subsequent to passage through the ultrasound field (acoustic aberrator present) as computed via 40 iterations of the phase retrieval algorithm. Views are shown for angles 0, 21.6, 43.2, 64.8, 86.4, 108, 129.6, 151.2, and 172.8 degrees corresponding to the diffraction plane optical intensity shown in Figure A.12. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. Figure A.13 shows the corresponding views for the experiment with no aberrator affixed to the emitting surface of the transducer.
Figure A.15: Radon transforms of several planar slices through the ultrasound field for the unaberrated case. Each level has a normal approximately parallel to the emitting surface of the transducer. Levels corresponding to rows 17, 34, 51, 68, 85, 102, 119, and 136 in the views shown in Figure A.13 are shown beginning with the upper left and proceeding across the rows. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. Corresponding reconstructions are shown in Figures 6.17 and 6.19.
Figure A.16: Radon transforms of several planar slices through the ultrasound field for the aberrated case. Each level has a normal approximately parallel to the emitting surface of the transducer. Levels corresponding to rows 37, 54, 71, 88, 105, 122, 139, and 156 in the views shown in Figure A.14 are shown beginning with the upper left and proceeding across the rows. A different linear gray scale is used in each image to display the wide dynamic range of the data without distorting feature shape. Corresponding reconstructions are shown in Figures 6.18 and 6.20.
Consider a Fredholm integral equation for a shift-invariant kernel $K(r)$

$$f(r) = \int_{-\infty}^{\infty} ds \, K(r - s)g(s). \quad (B.1)$$

Computing the ray sum of $f(r)$ in Eq. (B.1) parallel to the $r_n$ axis yields,

$$\int_{-\infty}^{\infty} dr_n \, f(r) = \int_{-\infty}^{\infty} dr_n \, \int_{-\infty}^{\infty} ds \, K(r - s)g(s) \quad (B.2)$$

$$= \int_{-\infty}^{\infty} ds \, g(s) \int_{-\infty}^{\infty} dr_n \, K(r_n - s_n, r' - s'), \quad (B.3)$$

where $r'$ and $s'$ are vectors containing all variables in $r$ and $s$ respectively except the $n$th. Now denote

$$K_n(r' - s') = \int_{-\infty}^{\infty} dr_n \, K(r_n - s_n, r' - s') = \int_{-\infty}^{\infty} du K(u, r' - s'). \quad (B.4)$$

Defining

$$f_n(r') \equiv \int_{-\infty}^{\infty} dr_n \, f(r) \quad (B.5)$$

and

$$g_n(s') \equiv \int_{-\infty}^{\infty} ds_n \, g(s) \quad (B.6)$$

we have

$$f_n(r') = \int_{-\infty}^{\infty} ds \, K_n(r' - s')g(s)$$

$$= \int_{-\infty}^{\infty} ds' \, K_n(r' - s') \int_{-\infty}^{\infty} ds_n \, g(s) \quad (B.7)$$

$$= \int_{-\infty}^{\infty} ds' \, K_n(r' - s') g_n(s').$$

Equation (B.7) states that the ray sum of the output of an $m$-dimensional (for $m > 1$) linear, shift-invariant system may be computed from the ray sums of the kernel and input functions. Clearly, the
Fourier rotation and convolution theorems indicate that the entire Radon transform of the output function may be computed in this fashion. It should be noted that if the kernel is rotationally invariant about the origin (as it is in the case of wave propagation) then the reduced kernel $K_n(r' - s')$ is the same for every angle. This is very useful as it allows us to compute the Radon transform of a three-dimensional acoustic field from the ray sum of the aperture function. It also implies that we may propagate the ray sum of a measured field to a different location and obtain the ray sum of the three-dimensional field at the new location.

For the case of Fresnel propagation we may explicitly see the relationship between the ray sums kernel, input (aperture), and output (propagated pulse) as follows. Let $u_2(r_2)$ be the complex amplitude of the field over the plane at $z = z_2$ generated by the field $u_1(r_1)$ in plane $z = z_1$ and let $z_{21} = z_2 - z_1$. \(^1\) Then we may abbreviate the effect of the Fresnel diffraction integral

$$u_2(r_2) = \frac{\exp(jkz_{21})}{j\lambda z_{21}} \int \exp \left( \frac{jk(r_2 - r_1)^2}{2z_{21}} \right) u_1(r_1) \, dr_1. \quad (B.8)$$

We wish to compute the line integral of $u_2(r_2)$. In general

$$r_i = x_i \hat{x} + y_i \hat{y}, \quad (B.9)$$

where $i = 1, 2$ specifies the plane. Making the term $(r_2 - r_1)^2$ explicit gives

$$u_2(r_2) = \frac{\exp(jkz_{21})}{j\lambda z_{21}} \int \exp \left( \frac{jk(x_1 - x_2)^2 + (y_1 - y_2)^2}{2z_{21}} \right) u_1(r_1) \, dr_1, \quad (B.10)$$

Without loss of generality we may choose to integrate the complex field $u_2(r_2)$ along the $x$-direction. Integrating both sides of Eq. (B.10) with respect to $x_1$ yields the following expression for $u_2'(y_2)$, the integrated field

$$u_2'(y_2) = \frac{\exp(jkz_{21})}{\sqrt{j\lambda z_{21}}} \int \exp \left( \frac{jk(y_1 - y_2)^2}{2z_{21}} \right) u_1(r_1) \, dy_1, \quad (B.11)$$

where we have used Saff and Snider, page 265 [171]

$$\int_{-\infty}^{\infty} \exp(ia^2x^2) \, dx = \sqrt{i\pi/a} \quad (B.12)$$

to infer that

$$\int_{-\infty}^{\infty} \exp \left( \frac{jk(x_1 - x_2)^2}{2z_{21}} \right) \, dx = \sqrt{j\lambda z_{21}}. \quad (B.13)$$

We now note that Eq. (B.11) is a double integral and that the remaining one-dimensional Fresnel transform kernel is a function of only one coordinate. Thus we may write

$$u_2'(y_2) = \frac{\exp(jkz_{21})}{\sqrt{j\lambda z_{21}}} \int \exp \left( \frac{jk(y_1 - y_2)^2}{2z_{21}} \right) u_1'(y_1) \, dy_1, \quad (B.14)$$

where $u_1'(y_1)$ is the line integral along the $x$-dimension of the aperture field.

---

\(^1\)We shall in general denote distance between two planes $p_i$ and $p_j$, with respective $z$-coordinates $z_i$ and $z_j$, as $z_{ij} = z_i - z_j$. 

*Optical Measurement of Wide Bandwidth Ultrasound Fields*
MEASUREMENT SYSTEM MODEL

Consider the system shown in Figure 2.1. We wish to relate the optical intensity in plane 1 to that impinging on the recording or CCD plane. Figure C.1 shows the idealized two lens system we will use to build our model.

Using the operator notation described in Appendix D we may write the following by inspection of Figure C.1.

\[
  u_5(r_5) = D[z_{54}] R[z_{54}] Q \left[ -\frac{1}{f_2} \right] D[z_{43}] R[z_{43}] D[f_1] R[f_1] Q \left[ -\frac{1}{f_1} \right] D[z_{21}] R[z_{21}] u_1(r_1)
\]

(C.1)

Gathering linear phase factors we have

\[
  u_5(r_5) = D[z_{51}] R[z_{54}] Q \left[ -\frac{1}{f_2} \right] R[z_{43}] R[f_1] Q \left[ -\frac{1}{f_1} \right] R[z_{21}] u_1(r_1).
\]

(C.2)

Thus the complex amplitude of the field in plane \( p_1 \) may be related to that in plane \( p_5 \) via

\[
  u_5(r_5) = D[z_{51}] T_{51} u_1(r_1).
\]

(C.3)

Figure C.1: Idealized two lens imaging system. Plane \( p_1 \) is imaged onto plane \( p_5 \) (the CCD sensing plane). In the experiments described here the focal lengths of lenses \( \ell_1 \) and \( \ell_2 \) are \( f_1 = 580 \text{ mm} \) and \( f_2 = 135 \text{ mm} \) respectively.
We first examine the operator $T_{31}$ which relates $u_3(r_3)$ and $u_1(r_1)$. Note
\[ u_3(r_3) = D_{[31]} T_{31} u_1(r_1) = D_{[31]} R_{[f_1]} Q \left[ -\frac{1}{f_1} \right] R_{[z_{21}]} u_1(r_1). \]

Now
\[
T_{31} = R_{[f_1]} Q \left[ -\frac{1}{f_1} \right] R_{[z_{21}]}
\]
\[= R_{[f_1]} Q \left[ -\frac{1}{f_1} \right] F^{-1} Q \left[ -\lambda^2 z_{21} \right] F
\]
\[= \frac{1}{j} Q \left[ \frac{1}{f_1} \right] V \left[ \frac{1}{\lambda f_1} \right] Q \left[ -\lambda^2 z_{21} \right] F
\]
\[= \frac{1}{j} Q \left[ \frac{1}{f_1} \right] Q \left[ 1 - \frac{z_{21}}{f_1} \right] V \left[ \frac{1}{\lambda f_1} \right] F
\]
\[= \frac{1}{j} Q \left[ 1 - \frac{z_{21}/f_1}{f_1} \right] V \left[ \frac{1}{\lambda f_1} \right] F.\] (C.8)

Examining $T_{51}$ we note that
\[
T_{51} = R_{[z_{34}]} Q \left[ -\frac{1}{f_2} \right] R_{[z_{43}]} T_{31}
\]
\[= \frac{1}{j} R_{[z_{34}]} Q \left[ -\frac{1}{f_2} \right] R_{[z_{43}]} Q \left[ 1 - \frac{z_{21}/f_1}{f_1} \right] V \left[ \frac{1}{\lambda f_1} \right] F
\]
\[= \frac{1}{j} R_{[z_{34}]} Q \left[ \frac{1}{z_{43}} - \frac{1}{f_2} \right] V \left[ \frac{1}{\lambda z_{43}} \right] F Q \left[ \frac{1}{z_{43}} + \frac{1 - z_{21}/f_1}{f_1} \right] V \left[ \frac{1}{\lambda f_1} \right] F
\]

Expanding $R_{[z_{34}]}$ gives
\[= -Q \left[ \frac{1}{z_{34}} \right] V \left[ \frac{1}{\lambda z_{34}} \right] F Q \left[ \frac{1}{z_{43}} + \frac{1 - z_{21}/f_1}{f_1} \right] V \left[ \frac{1}{\lambda f_1} \right] F
\]

Moving $V[1/\lambda z_{34}]$ past $F$ and $Q[1/z_{34} + 1/z_{43} - 1/f_2]$ gives
\[
T_{51} = -Q \left[ \frac{1}{z_{34}} \right] F Q \left( \lambda z_{34} \right)^2 \left[ \frac{1}{z_{43}} + \frac{1 - z_{21}/f_1}{f_1} \right] V \left[ \frac{z_{54}}{z_{43}} \right] F
\]
\[Q \left[ \frac{1}{z_{43}} + \frac{1 - z_{21}/f_1}{f_1} \right] V \left[ \frac{1}{\lambda f_1} \right] F.\] (C.9)

Using identity (D.25) yields
\[
T_{51} = -Q \left[ \frac{1}{z_{34}} \right] R \left[ \frac{2}{z_{54}} \left( \frac{1}{z_{54}} + \frac{1 - z_{43}}{f_2} \right) \right] F V \left[ \frac{z_{54}}{z_{43}} \right] F
\]
\[Q \left[ \frac{1}{z_{43}} + \frac{1 - z_{21}/f_1}{f_1} \right] V \left[ \frac{1}{\lambda f_1} \right] F.\] (C.10)

\[
T_{51} = -Q \left[ \frac{1}{z_{34}} \right] R \left[ \frac{2}{z_{54}} \left( \frac{1}{z_{54}} + \frac{1 - z_{43}}{f_2} \right) \right] V \left[ \frac{z_{43}}{z_{54}} \right] F
\]
\[Q \left[ \frac{1}{z_{43}} + \frac{1 - z_{21}/f_1}{f_1} \right] V \left[ \frac{1}{\lambda f_1} \right] F.\] (C.11)
Expanding $R \left[ z_{54}^2 (1/z_{54} + 1/z_{43} - 1/f_2) \right]$ and setting $h = 1/z_{54} + 1/z_{43} - 1/f_2$ brings

$$T_{51} = -Q \left[ \frac{1}{z_{54}} + \frac{1}{z_{54}^2 h} \right] V \left[ \frac{1}{\lambda z_{54}^2 h} \right] FQ \left[ \frac{1}{z_{54}} + \frac{1}{z_{43}^2} \right] V \left[ \frac{1}{z_{43}} + \frac{1 - z_{21}/f_1}{f_1} \right] V \left[ \frac{1}{\lambda f_1} \right] F. \quad (C.12)$$

Moving $Q \left[ 1/z_{54}^2 h \right]$ past $V \left[ -z_{43}/z_{54} \right]$ gives

$$T_{51} = -Q \left[ \frac{1}{z_{54}} + \frac{1}{z_{54}^2 h} \right] V \left[ \frac{1}{\lambda z_{54}^2 h} \right] FV \left[ \frac{z_{43}}{z_{54}} \right] Q \left[ \frac{1}{z_{43}} + \frac{1 - z_{21}/f_1}{f_1} - \frac{1}{z_{43} z_{54} h} \right] V \left[ \frac{1}{\lambda f_1} \right] F. \quad (C.13)$$

Moving $F$ past $V \left[ -z_{43}/z_{54} \right]$ gives

$$T_{51} = -Q \left[ \frac{1}{z_{54}} + \frac{1}{z_{54}^2 h} \right] V \left[ \frac{1}{\lambda z_{43} z_{54} h} \right] FQ \left[ \frac{1}{z_{43}} + \frac{1 - z_{21}/f_1}{f_1} - \frac{1}{z_{43} z_{54} h} \right] V \left[ \frac{1}{\lambda f_1} \right] F. \quad (C.14)$$

We note that under the following condition the complex amplitude of the field in plane $p_5$ will be a scaled image of that in plane $p_1$ multiplied by a quadratic phase factor.

$$z_{21} = -\frac{f_1 (f_2 z_{52}^2 - z_{54} z_{52} + z_{54}^2)}{f_2 f_1 - z_{54} f_1 - f_2 z_{52} + z_{54} z_{52} - z_{54}^2} \quad (C.15)$$

We have made the substitution $z_{43} = z_{52} - z_{54} - f_1$ to arrive at Eq. (C.15) in order to express the fact that plane $p_4$ is the only movable plane in our system. We may denote an operator relating the complex field amplitudes in planes $p_1$ and $p_5$ under this imaging condition as

$$T_{51}^i = -Q \left[ \frac{1}{z_{54}} + \frac{1}{z_{54}^2 h} \right] V \left[ \frac{1}{\lambda z_{43} z_{54} h} \right] FQ \left[ \frac{1}{z_{43}} + \frac{1 - z_{21}/f_1}{f_1} - \frac{1}{z_{43} z_{54} h} \right] V \left[ \frac{1}{\lambda f_1} \right] F. \quad (C.16)$$

In order to calculate the actual scale factor we move $F$ past $V \left[ 1/\lambda f_1 \right]$ and obtain

$$T_{51}^i = -Q \left[ \frac{1}{z_{54}} + \frac{1}{z_{54}^2 h} \right] V \left[ \frac{1}{\lambda z_{43} z_{54} h} \right] F \left[ \frac{f_2 f_1}{z_{43} z_{54} (1/z_{54} + 1/z_{43} - 1/f_2)} \right]. \quad (C.17)$$

giving a scale factor of

$$M = \frac{f_2 f_1}{f_2 f_1 - z_{54} f_1 - f_2 z_{52} + z_{54} z_{52} - z_{54}^2}, \quad (C.18)$$

where we have again made the substitution $z_{43} = z_{52} - z_{54} - f_1$.

In our system we must take into account the fact that the space between planes $p_1$ and $p_2$ is filled with water, for which the refractive index $n \approx 1.32$. In order to understand how to modify Eq. (C.15) to take this into account we look at propagation in the Fourier domain. There, we have

$$U(k^2, z_2) = e^{ik_2 z_2} U(k^2, z_1) = e^{i z_2 \sqrt{k^2 - k_\perp^2}} U(k^2, z_1), \quad (C.19)$$

where $k = \omega/c$ is the wave number, $k_\perp = \sqrt{k^2 + k_\parallel^2}$, and $k_\parallel$ and $k_\perp$ are Fourier variables. If our refractive index in $n$ we may write $k = nk_\circ$, where $k_\circ$ is the free-space wave number, and thus

$$U(k_{\perp}, z_2) = e^{ik_\parallel z_2} \sqrt{1 - (k_\parallel/k_\circ)^2} U(k_{\perp}, z_1). \quad (C.20)$$
Under a Fresnel approximation

\[ U(k_\perp, z_2) = e^{ik_0(nz_{21})}e^{-i(k_\perp^2/2k_0)(z_{21}/n)}U(k_\perp, z_1) \]  \hspace{1cm} (C.21)

so that the distance for the uniform phase delay is increased by a factor of \( n \) and the distance describing diffraction is reduced by a factor of \( n \). We may interpret this by saying the same amount of diffraction occurs in less distance in free space. Thus the physical \( z_{21} \) we measure in our system is related to what we would measure in free-space by \( z_{21}^{\text{free}} = z_{21}/n \) and we may write the imaging condition for our system as

\[ z_{21} = \frac{nf_1}{f_2f_1 - z_{54}z_{52} + z_{54}^2} \left( f_2z_{52} - z_{54}z_{52} + z_{54}^2 \right). \]  \hspace{1cm} (C.22)

In deriving the system model of Appendix C an operator notation is used to facilitate the manipulations. This notation is described in the literature [172]. A minor variation on this description is given here for completeness and clarity of the derivation in Appendix C. Section D.1 defines the operators themselves and discusses some basic properties (particularly with respect to the effect of the operators on the units of a quantity). Identities used in Appendix C are given in Sec. D.2.

### D.1 Definition of Operators

We describe here a slightly modified version of the operator notation given by Nazarathy [172]. Particular attention must be paid to the parameters of the operators. Each operator operates on the variables present in the function to its right. If a function has variables with units \((m^{-1})\) then the operator on its immediate left will operate on this variable. If a function has variables with units \((m)\) then the operator on its immediate left will operate on \(these\) variables. This means the required units for the parameters to an operator are dependent on the function immediately to its right. As an example the Fresnel propagation operator \(R [s]\) has a parameter with units of \((m)\) when it operates on a function whose transverse coordinates are in meters. However, when \(R [s]\) operates on a function whose transverse coordinates are in inverse meters then \(s\) has units of inverse cubic meters. The scaling operator has a unique effect on units. When it is applied to a function it changes the units of the function’s variables. Define the transversal radius vector \(r\) as

\[
r = x \hat{x} + y \hat{y}.
\]  

(D.1)

Let \(u_2(r_2)\) be the complex amplitude of the field over the plane at \(z = z_2\) generated by the field \(u_1(r_1)\) in plane \(z = z_1\) and let \(z_{21} = z_2 - z_1\). Then we may abbreviate the effect of the Fresnel diffraction integral

\[
u_2(r_2) = \frac{\exp j k z_{21}}{j \lambda z_{21}} \int \exp \left(\frac{j k (r_2 - r_1)^2}{2 z_{21}}\right) u_1(r_1) dr_1,
\]  

(D.2)

where the integral is over the entire real line, as

\[
u_2 = \exp(j k z_{21}) R [z_{21}] u_1.
\]  

(D.3)

\(^1\)We shall in general denote distance between two planes \(p_i\) and \(p_j\), with respective \(z\)-coordinates \(z_i\) and \(z_j\), as \(z_{ij} = z_i - z_j\).
Operator Identities

We define the Fourier transform operator as

$$\mathbf{F} f(r) = \int f(r) \exp(-j2\pi \mathbf{v} \cdot \mathbf{r})d\mathbf{r}, \quad (D.4)$$

where the integral is over the entire real line. Note that the variable of integration is specified by the operand and not the operator. Scaling and quadratic phase multiplication operators are defined respectively as

$$V[s] f(r) = jsf(sr), \quad (D.5)$$

and

$$Q[s] f(r) = \exp(jksr^2/2)f(r), \quad (D.6)$$

where $r = |r|$. Lastly we define a phase delay operator by

$$D[s] f(r) = \exp(jks)f(r). \quad (D.7)$$

By expanding Eq. (D.2) we may directly write down the following identity.

$$R[z_21] = \frac{1}{j} Q[z_21] V[\frac{1}{\lambda z_{21}}] F Q[\frac{1}{z_{21}}] \quad (D.8)$$

Note that we have used the fact that $\lambda z_{21}$ is always positive to comply with the precise definition of $V[s]$. Recognizing Eq. (D.2) as a convolution we may write

$$u_2(r_2) = \frac{1}{j\lambda z_{21}} D[z_{21}] \exp \left( jkr^2/2z_{21} \right) \otimes u_1(r_1). \quad (D.9)$$

Operating on both sides with $\mathbf{F}$ and applying the modulation-convolution theorem yields

$$\mathbf{F} u_2(r_2) = \frac{1}{j\lambda z_{21}} D[z_{21}] \mathbf{F} \exp \left( jkr^2/2z_{21} \right) \mathbf{F} u_1(r_1). \quad (D.10)$$

Noting that

$$\mathbf{F} \exp \left( jkr^2/2z_{21} \right) = j|\lambda z_{21}| \exp \left( -jkf^2\lambda^2 z_{21}/2 \right) \mathbf{F} u_1(r_1), \quad (D.11)$$

where $f = f_x \hat{x} + f_y \hat{y}$, we apply $\mathbf{F}^{-1}$ to both sides of Eq. (D.10) to obtain

$$u_2(r_2) = D[z_{21}] \mathbf{F}^{-1} Q[-\lambda^2 z_{21}] \mathbf{F} u_1(r_1) \quad (D.12)$$

We now have the expansion

$$R[z] = \mathbf{F}^{-1} Q[-\lambda^2 z] \mathbf{F}. \quad (D.13)$$

D.2 Operator Identities

The following identities follow directly from the definitions or may be easily derived. Note that $s_i$ refers to a generic coordinate (any of $x$, $y$, or $z$) while $z$ specifically refers to the coordinate in the
direction of propagation (non-transverse). Naturally, $s_i$ is always real.

$$Q[s_1]Q[s_2] = Q[s_1 + s_2] \quad (D.14)$$
$$Q^{-1}[s] = Q[-s] \quad (D.15)$$
$$Q[0] = 1 \quad (D.16)$$
$$D[s_1]Q[s_2] = Q[s_2]D[s_1] \quad (D.17)$$
$$D[s_1]V[s_2] = V[s_2]D[s_1] \quad (D.18)$$
$$D[z]F = FD[z] \quad (D.19)$$
$$D[z]R[z] = R[z]D[z] \quad (D.20)$$
$$D[s_1]D[s_2] = D[s_1 + s_2] \quad (D.21)$$
$$V[s_1]Q[s_2] = Q[s_1^2 s_2]V[s_1] \quad (D.22)$$
$$FF = V[-1] \quad (D.23)$$
$$V[s]F = FV[1/s] \quad (D.24)$$
$$FQ[s] = R[-s/\lambda^2]F \quad (D.25)$$

Proof that $FQ[s] = R[-s/\lambda^2]F$ may be seen as follows.

$$FQ[s]g(r) = \int dr \exp(-j2\pi\nu \cdot r) \exp(j\pi(s/\lambda) r^2)g(r)$$
$$= \frac{j\lambda}{s} \exp(-j\pi\nu^2/s) \otimes G(\nu)$$
$$= \frac{j\lambda}{s} \int d\tau \exp(-j\pi\lambda(\nu - \tau)^2/s)G(\tau) \quad (D.26)$$
$$= \frac{1}{j\lambda(-s/\lambda^2)} \int d\tau \exp \left(\frac{jk(\nu - \tau)^2}{2(-s/\lambda^2)}\right)G(\tau)$$
$$= R[-s/\lambda^2]Fg(r)$$


**IMAC** Image Acquisition. An IMAC board samples the analog data segment of an NTSC video signal and converts the samples to digital form. For example an 8-bit IMAC board delivers sample values between 0 and 255. The transformation performed on the signal may be affine as opposed to linear.

**NTSC** National Television Standards Committee.

**CCD** Charge Coupled Device. A semiconductor device used here as an optical detector. CCD cameras consist principally of two-dimensional arrays of CCD elements. Accumulation of electronic charge in potential wells under each element occurs proportionally to incident photon count. The charge under each element is read out in a raster fashion.

**SDE** Summed Distance Error. The SDE is an error metric expressing the distance of a point $x$ from two closed sets. Denoting projection operators onto each set by $P_1$ and $P_2$, the SDE is defined as

$$J(x) \triangleq \| P_1 x - x \| + \| P_2 x - x \|.$$

**MGP** Method of Generalized Projections. The MGP is an iterative technique for the solution of a general (possibly nonlinear) set of equations. Constraints on the solution are described via general (possibly non-convex) closed sets. A solution to the problem is required to be a member of each of the sets. The method requires definition of a projection operator for each set that restricts its operand uniquely to a set member. Beginning with a point satisfying one of the constraints, possible solutions are successively projected onto each set. For the case of two sets the algorithm possesses a monotonic error reduction property defined in terms of the Summed Distance Error (SDE).

**POCS** Projection Onto Convex Sets. The method of POCS is an iterative technique for finding the solution to a problem known to lie in the intersection of a number of mathematically convex sets. The method requires definition of a projection operator for each set that maps a point (possible solution) to the closest member of the set. Beginning with a point in one of the sets the projection operators are successively applied (composed). A type of convergence is guaranteed for a broad range of problem parameters.

**FSR** Fourier Slice Reconstruction. The FSR theorem states that the one-dimensional Fourier transform of the projection of (or ray sum through) an object function is equal to a slice (the value along a line) of the two-dimensional Fourier transform of the object function. Rotation of the object function about the origin rotates the Fourier transform of the object through an identical angle in the frequency domain providing a means of determining the complete two-dimensional Fourier transform of the object function and hence the object function itself.
LIST OF SYMBOLS

\( J \)
Jacobian matrix, see equation (2.50), page 22

\( J_n(f_n) \)
Summed Distance Error (SDE) at \( n \)th iteration, see equation (3.47), page 45

\( R(z) \)
z-transform of discrete autocorrelation function, see equation (3.15), page 39

\( C(n) \)
Set of all complex, \( n \)-dimensional sequences with finite causal support, page 38

\( C(n, k) \)
Set of all polynomials of degree \( n \) in \( k \) variables with complex coefficients, page 41

\( \hat{X}(z) \)
z-transform of the conjugate space-reversal of \( x(n) \), see equation (3.12), page 38

\( \alpha_e \)
Fresnel Taylor’s series phase error (Fourier domain), see equation (2.32), page 16

\( \beta_e \)
Fresnel Taylor’s series phase error (spatial domain), see equation (2.37), page 17

\( \chi \)
Electric susceptibility, see equation (2.39), page 18

\( \delta n(r) \)
Small refractive index perturbation, see equation (3.31), page 42

\( \ell_2 \)
Optical system lens (see Figure 2.1), page 11

\( \ell_3 \)
Optical system lens (see Figure 2.1), page 11

\( \varepsilon \)
Permittivity (farad/meter), see equation (2.7), page 12

\( \varepsilon_0 \)
Permittivity of free space, \( \varepsilon_0 \approx 8.85 \times 10^{-12} \) farads/meter, see equation (2.39), page 18

\( \varepsilon_r \)
Relative permittivity, see equation (2.40), page 18

\( \lambda \)
Optical wavelength (meters), see equation (2.33), page 16

\( \ln f_s \)
Natural logarithm of \( f_s \), see equation (2.24), page 15

\( \mu \)
Permeability (henry/meter), see equation (2.7), page 12

\( \nabla \cdot \mathbf{A} \)
Divergence of \( \mathbf{A} \), see equation (2.5), page 12

\( \nabla \nabla \cdot \mathbf{A} \)
Vector divergence of \( \mathbf{A} \), see equation (2.8), page 13

\( \nabla \times \mathbf{A} \)
Curl of \( \mathbf{A} \), see equation (2.7), page 13

\( \nabla^2 \mathbf{A} \)
Laplacian of \( \mathbf{A} \), see equation (2.9), page 13

\( \nu \)
Temporal frequency (cycles/second), see equation (2.43), page 20

\( \omega \)
Angular or radian frequency (radians/second), see equation (2.27), page 16

\( \left( \frac{\partial n}{\partial p} \right)_s \)
Piezooptic coefficient (atm\(^{-1}\)), see equation (2.25), page 15

\( \prod_{n=1}^{N} a_n \)
Product of \( a_n \) for \( n = 1, 2, \ldots, N \), see equation (3.11), page 38

\( \mathbb{P}_i \)
Projection onto set \( C_i \), see equation (3.35), page 44

\( \mathcal{B} \)
Magnetic flux density (webers/meter\(^2\)), see equation (2.4), page 12

\( \mathcal{D} \)
Electric flux density (coulombs/meter\(^2\)), see equation (2.4), page 12

\( \mathcal{E} \)
Electric field strength (volts/meter), see equation (2.4), page 12

\( \mathcal{H} \)
Magnetic field strength (amperes/meter), see equation (2.4), page 12

\( \mathbf{A} \)
Arbitrary vector function, see equation (2.10), page 13

\( \mathcal{J} \)
Electric current density (amperes/meter\(^2\)), see equation (2.4), page 12
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