Image formation in vibro-acoustography

Abstract

In this work, the beamforming and image formation of vibro-acoustography systems are studied. Vibro-acoustography is an imaging technique that maps the mechanical response of an object (or biological tissue) to a localized dynamic ultrasound radiation force. This force is generated by a focused modulated ultrasound beam. Two schemes are used for this purpose: dual and amplitude-modulated single beam modes. The former uses two focused ultrasound beams at slightly different frequencies. The latter uses a focused modulated ultrasound beam whose modulation frequency is much smaller than the carrier frequency. In response to the dynamic radiation force, the object vibrates emitting an acoustic field which can be detected some distance way by a microphone or hydrophone. The detected signal is used to synthesize the image of the object. Dual beam mode presents superior results for vibro-acoustographic images. The point-spread function (PSF) of the system depends on the dynamic radiation force on a point-target in the focal zone of the transducer. Stemming from the Brillouin radiation-stress tensor, the ultrasound radiation force exerted by modulated waves on a point-target is calculated by means of acoustic scattering theory. This formulation generalizes the collimated quasi-plane wave model. Both transverse and axial components of the radiation force can be assessed in this formulation. It is shown that the collimated quasi-plane wave model is still a good approximation for beamforming in vibro-acoustography. Effects of the transverse component of the radiation force in the spatial resolution of system is discussed. Three different transducers are analyzed with the radiation force formulation proposed here: confocal and sector spherical transducers and linear array transducers. The first two transducers are studied based on analytic results for the acoustic potential on the transducer focal plane. Measurements of the amplitude of the axial and transverse components of radiation force produced by the confocal transducer are in good agreement with theoretical results. Ultrasound waves radiated by linear array transducers are computationally simulated based on the spatial impulse method. More emphasis is given to linear array beamforming due to the potential use of these transducers for clinical applications.
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Chapter 1

Introduction

The use of ultrasound in medical imaging and nondestructive material evaluation is a well-established joint-venture of science and technology. The pulse-echo technique is used for real-time gray scale imaging. Several specialized methods have been proposed and investigated such as 3D-ultrasound imaging, tissue harmonic imaging, contrast agents, and elasticity imaging. The focus of this thesis is on a technique called vibro-acoustography which is a modality of elasticity imaging. In this introduction, vibro-acoustography is presented in the context of elasticity imaging. Several concepts associated with vibro-acoustography such as acoustic radiation force and acoustic emission are discussed. This dissertation can be summarized as follows:

- Chapter 1 gives an overview of vibro-acoustography and its framework. Several applications of vibro-acoustography are discussed.
- Chapter 2 presents the background needed to develop this work. Some attention is given to the mathematical background. Key concepts of vibro-acoustography system are explored in more detail.
- Chapter 3 presents the theory of ultrasound radiation force produced by any kind of beam in the three-dimensional space.
- Chapter 4 studies the effects of the axial and transverse components of radiation force in the system point-spread function.
- Chapter 5 studies the beamforming of linear array transducers in vibro-acoustography through computational simulations.
- Chapter 6 summarizes this work and discusses important aspects presented here.
Chapter 1: Introduction

1.1 Elasticity imaging

Changes in elasticity of soft tissues are often related to pathology. For example, some tumors of the breast are detected by palpation through the overlying tissue. In palpation a force is exerted on the body surface and it spreads over the tissues. If the response of a tissue is sufficiently different from the surrounding tissues, the physician may identify the tissue as abnormal. Elastic constants are related to thermodynamic properties of materials and they can be associated to a variety of physical parameters. In soft tissue, elastic properties such as shear and Young’s (stiffness) moduli may vary four orders of magnitude. Furthermore, these quantities can change by 1000-fold within the same tissue during such processes as tumor development or muscle contraction (as reported in Ref. [1]).

Most soft tissues are composed by 70 to 80% of water, while the remaining ingredients consist of various organic and inorganic compounds with low molecular weight. The variability of soft tissue density is small. The speed of sound, which is related to the bulk modulus, of most soft tissues may vary less than 10% [2]. Imaging methods based on bulk modulus variation usually present low image contrast when applied to soft tissues. For instance, methods based on x-ray, which depend upon the radiation absorption related to the material density, have low contrast in soft tissues. An imaging technique able to assess elastic properties of tissues may be applied in a new set of medical diagnostic problems.

Elasticity imaging [3] is the general field of quantitative methods to image elastic properties of tissues. The objective of elasticity imaging is to map tissue elastic properties in an anatomically meaningful picture to provide useful diagnostic information. The methods use an external source of force to produce a static or dynamic stress distribution on the probed tissue. The applied stress causes a displacement distribution within the tissue which can be measured by nuclear magnetic resonance, ultrasound, or optical methods. Muthupillai et al. [4] developed magnetic resonance elastography, which consists of visualizing the propagation of shear waves in a material (or tissue) with the magnetic resonance imaging (MRI) technique. The shear waves are generated by an external source of force. The external force produces a spatially wide stress distribution inside the object or tissue. Analysis of the response of the region of interest within the object can be complicated due to the complexity of the stress distribution in the object.

An alternative approach is to use a localized stress distribution directly in the region of interest. The use of localized acoustic radiation force to assess tissue properties (hardness) was first proposed by Sugimoto et al. [5]. In this method, an impulsive radiation force is exerted on a localized region in the tissue by a focused ultrasound pulsed beam. The force causes a displacement which is measured by ultrasound pulse-echo techniques. Acoustic radiation force produced by a modulated ultrasound beam has been used to generate shear elastic waves in tissue as an imaging method [6]. In this method, the resulting shear waves due to the radiation force are generated in the tissue at the
modulation frequency. The shear waves are detected by an imaging transducer. Walker et al. [7] used pulsating acoustic radiation force to produce images of viscoelastic parameters in a mimicking gel phantom. Nightingale et al. [8] established the acoustic radiation force impulsive imaging (ARFI) in which a single transducer is used to generate the localized radiation force and measure the resulting displacement by means of ultrasound correlation-based methods. Fatemi et al. [9] proposed vibro-acoustography as an imaging method which is based on localized dynamic ultrasound radiation force exerted in tissue (or object) by an ultrasound modulated beam.

1.2 What is vibro-acoustography?

When an object is tapped, it emits a sound, which basically depends upon its mechanical properties. Tapping an object and hearing the emitted sound is a way to assess its mechanical properties. In medicine, physicians customarily use palpation to recognize abnormal tissues. The utility of palpation is limited to the size and location of the abnormal tissue. If the abnormality is too small to be sensed by touch, or if it lies deep in the body, practically nothing can be inferred by palpation.

Ultrasound vibro-acoustography is an imaging technique that produces a map of the mechanical response of an object to a localized dynamic force. This technique is based on the dynamic ultrasound radiation force generated by a modulated ultrasound beam. The dynamic force causes a vibration in the object (tissue) which emits an acoustic field that can be detected by a sensitive hydrophone or microphone some distance away. The object response to the dynamic radiation force is called acoustic emission. There are two basic ways to produce dynamic ultrasound radiation force: amplitude modulated (AM) single beam and dual beam. The former mode uses an AM ultrasound beam to produce the radiation force. This configuration can be accomplished with a focused transducer driven by a modulated signal. The latter mode requires two ultrasound beams intersecting in space. Hence, the resulting ultrasound beam is modulated only in the intersecting region. This config-
Figure 1.2: Driving signals in vibro-acoustography. (a) and (b) Continuous-wave (CW) signals at frequencies $f_a = 3 \text{ MHz}$ and $f_b = 3.01 \text{ MHz}$. (b) Modulated ultrasound field whose modulation frequency is $\Delta f = 10 \text{ kHz}$. (d) Modulated ultrasound with tone burst (dotted line) with 150 ms of pulse duration.

duration requires two transducers to produce the ultrasound beams. Experiments show that dual beam mode is superior to single beam mode. This is because in single beam mode, the radiation force is produced along the wave path from the transducer to the object target. Furthermore, the transducer itself produces an acoustic emission that interferes with the acoustic emission of the object.

In dual beam mode the modulated ultrasound beam is produced by using two continuous-wave (CW) or tone-burst signals at frequencies $f_a = f_0$ and $f_b = f_0 + \Delta f$ slightly different. The quantities $f_0$ and $\Delta f$ are the center and the modulation frequencies, respectively. The acoustic emission of the object occurs at the beat frequency $\Delta f$ of the ultrasound beams. In Figure 1.1, we have a description of a vibro-acoustography imaging system. A two-element confocal spherical transducer of 45 mm diameter, operating in dual beam mode, generates the modulated ultrasound beam in a water tank. Each element is driven by a CW-signal at frequencies $f_a = 3 \text{ MHz}$ and $f_b = 3.01 \text{ MHz}$. The driving frequencies depend upon the transducer characteristics. The ultrasound beams interfere in the focal zone of the transducer producing a dynamic radiation force at the beat frequency $\Delta f = 10 \text{ kHz}$ in the probed object. The frequency $\Delta f$ can be chosen from few Hertz to 100 kHz. Figure 1.2 shows the driving signals of the system. The dimensions of the focal zone of the confocal transducer is typically $(1.0 \times 1.0 \times 10.0) \text{ mm}$, laterally, in elevation, and axially. The resulting acoustic emission by the object is detected by a hydrophone some distance away from. The acoustic emitted field carries
information about local mechanical properties of the object as well as information of the object’s boundary conditions. In the described system, the imaging plane is defined as the transducer focal plane. The transducer mechanically scans an object in raster mode producing a two-dimensional image of the object. The *spatial resolution* or resolving power of the system, which accounts for the capacity to distinguish small and close objects, depends on the distribution of the dynamic radiation force in the focal zone of the transducer.

### 1.2.1 Image characteristics

Vibro-acoustography presents several attractive characteristics as an imaging method based on ultrasound. The most remarkable of these characteristics are listed below:

- **Speckless images:** A well-known artifact in ultrasound images is the speckle noise. This noise is produced by the interference of backscattered signals due to multiple scattering caused by small inhomogeneities (scatterers) in biological tissue. The size of these scatterers is of the order of the ultrasound wavelength. Images with speckle present a texture which has a more rapid spatial variation [10]. In vibro-acoustography, the detected signal (acoustic emission) has wavelength much larger than the tissue inhomogeneities dimensions. Consequently, the acoustic emission does not suffer any interference by multiple scattering. Hence, vibro-acoustographic images are free of speckle.

- **Assessment of dynamical properties of objects:** The application of a dynamic radiation force on an object allows the investigation of dynamical properties of the object. Resonances and different vibration modes can be analyzed by applying different modulation frequencies $\Delta f$, which can vary from few Hertz to 100 kHz. The analysis of dynamic properties may be useful for material characterization and evaluation.

- **High contrast images of hard inclusion in soft tissue:** Vibro-acoustographic images present good contrast and details of hard inclusions in soft tissue due to high acoustic impedance of the inclusion. The significant difference in the acoustic impedance compared to the soft tissue makes the hard inclusion a good ultrasound reflector which results in a stronger radiation force on the inclusion [11]. Furthermore, hard inclusions produce strong acoustic emission because it is a better radiation source compared to the surrounding tissue.

- **Beamforming similar to ultrasound systems:** For medical imaging application, ultrasound beams may be produced by different types of transducers such as annular flat, annular concave, or linear arrays. Vibro-acoustography beamforming shares similarities with its counterpart in conventional ultrasound (B-mode). However, some conceptual differences should be pointed
Figure 1.3: Images of a pig artery in vitro. (a), (b), and (c) Vibro-acoustographic images with difference frequency at 7, 24, and 41 kHz, respectively. (d) X-ray image with transversal photographies of the artery. (e) Conventional ultrasound image (C-scan).

out: (a) vibro-acoustographic images are generated by two intersecting ultrasound beams, while B-mode ultrasound uses only one beam; (b) the depth resolution of B-mode ultrasound depends on the transmitted pulse duration, in vibro-acoustography the depth resolution is related to how one incident beam intersects the other spatially; (c) vibro-acoustography uses narrow-band signals, while ultrasound systems are based on broad-band transmitted pulses; and (d) in most applications, ultrasound systems acquire data along one line in depth at a time (so-called A-line), while vibro-acoustography collects data from one point at a time. Despite these differences, the goals of beamforming for vibro-acoustography remain to achieve narrow beams with low sidelobes and minor effects of grating lobes.
1.2.2 State of the art

Vibro-acoustography has been used in a variety of medical imaging applications and material characterization. Here we list some of these applications.

- **Imaging of arteries and vessels:** One of the first applications of vibro-acoustography was to image arteries and vessels [9]. Figure 1.3 shows a pig artery scanned at $\Delta f = 7, 24,$ and $41$ kHz, with a calcified region as a bright spot, resembling its x-ray image. The choice of the difference frequency $\Delta f$ is arbitrary. However, images at higher frequencies exhibit more details. In this figure, we can compare vibro-acoustographic and conventional ultrasound images of the artery. To produce images corresponding to vibro-acoustographic images, the conventional ultrasound method had imaged the artery in the transversal plane or C-scan view. To obtain the C-scan image, the artery was scanned using the same transducer at $3$ MHz with the beam perpendicular to the transverse plane. The echos from the interface tissue-water overtook the reflected ultrasound waves due to the interior structures, including the calcification. This example shows the superiority of vibro-acoustography to conventional ultrasound applied to image calcification.

- **Imaging of heart valve leaflets:** Accumulation of calcium deposits in heart valves may interfere with the function of the leaflets to the point that the valve stops its normal functions. Early detection of calcium deposits on heart valve structures is important in diagnosing valve disease. Currently, no medical imaging method is reliable to detect calcium deposits on heart valve leaflets. Alizad *et al.* [12] used vibro-acoustography to detect calcium deposits on heart valve leaflets. The vibro-acoustographic images presented good correlation with x-ray images, indicating the potentials of vibro-acoustography for cardiac applications. However, *in vivo* application of vibro-acoustography for imaging moving structures, such as heart valves, requires further development.

- **Imaging of bones:** Callé *et al.* [13] investigated vibro-acoustography for imaging bone tissue. Variation of the acoustic emission amplitude with the tissue porosity was observed. Vibro-acoustographic images could be used to detect bone demineralization for diagnosis or prediction of osteoporosis.

- **Imaging with contrast agents:** Liquids that contain gas microbubbles (with typically $3 \, \mu\text{m}$ diameter) are used as contrast agents in conventional ultrasound to enhance images of blood flow. The sensitivity of vibro-acoustography to gas bubbles comes from the strong discontinuity in the acoustic impedance (variation of about $10^6$) in the interface medium bubble. Because radiation force depends on the acoustic impedance, the force exerted on the bubble is very strong. Belohlavek *et al.* [14] demonstrated experimentally that the acoustic emission by a
liquid flow with microbubbles varies linearly with the microbubble concentration up to the value of $3.6 \times 10^2$ bubbles/ml. Greenleaf et al. [15] showed that injected microbubbles in a femoral artery of a live pig changes the contrast of vibro-acoustographic images.

- **Imaging of microcalcification in breast tissue:** The detection of microcalcification in breast tissue can help early diagnosis of breast cancer. Currently x-ray mammography is the only clinical imaging technique used for detection of breast microcalcification. Pregnant and lactating women can present extremely dense breast tissue, which might change the sensitivity of the x-ray mammography. Furthermore, in these conditions, patients are relatively contraindicated to be submitted to x-ray radiation. Conventional ultrasound has been used to detect microcalcification with some limited results due to speckle noise [16]. Vibro-acoustography has been applied to image breast microcalcification [17]. The resulting vibro-acoustographic images are in agreement with corresponding x-ray mammography images of the specimens. The existence of microcalcification detected by vibro-acoustography is confirmed by histology. Microcalcification as small as 110 $\mu$m in diameter are detected in this method. The low performance of x-ray mammography to detect microcalcification in dense breast tissue may be overcome by using vibro-acoustography.

- **Monitoring tissue stiffness changing with temperature variation:** Konofagou et al. [18] has applied vibro-acoustography to estimate the stiffness of tissue after the application of high intensity focused ultrasound. Experimental results and simulations show that the resonance frequency of tissue vibration is shifted as a function of temperature.

- **Study of foreign bodies:** Images of solid foreign bodies such as metal inclusions can be easily imaged by vibro-acoustography. Some useful applications of this method are guiding catheters in body, imaging implants, and detecting foreign objects and fragments.

- **Quantitative methods:** Acoustic emission is related to the mechanical properties of the vibrating object. To quantitatively determine the mechanical parameters of an object it is necessary to have more information than just the acoustic emission. Additional information may be obtained from the frequency response of the object which can be assessed by sweeping the vibration frequency $\Delta f$ in the range of interest. This method is called vibro-acoustic spectrography. The shear viscosity of isopropyl alcohol versus temperature was measured using this method [9]. In this case, a tuning fork is immersed in the alcohol and its natural frequency is measured at different temperatures by vibro-acoustic spectrography. Based on these measured values, the change in the shear viscosity is determined. Chen et al. [19] established a method to measure shear modulus of gel phantoms by using a small embedded sphere as a target based on vibro-acoustic
spectrography. In this method, the vibration of the sphere is measured by a laser vibrometer or Doppler ultrasound.

1.3 Acoustic radiation force

To characterize a vibro-acoustography system, we need to understand how the ultrasound (acoustic) radiation force\(^1\) is generated on the probed object. Acoustic radiation force was probably first investigated by Rayleigh [20]. A vast discussion of acoustic radiation force can be seen in Refs. [21, 22, 23, 24] (and references therein).

It is well known that sound waves carry momentum. When a sound wave in a fluid strikes an object, part of its momentum becomes a force on the object. Furthermore, nonlinear properties of wave motion in the propagating medium also contributes to the radiation force on the object. In simple terms, acoustic radiation force is a time averaged force exerted by an acoustic field on an object or boundary surface. The physical processes leading to acoustic radiation force are quite complex. This force depends on the type of propagating medium (lossless or lossy fluids, elastic solids, and viscoelastic materials), mechanical properties and geometry of the target object.

1.3.1 Framework of acoustic radiation force

Two types of acoustic radiation force have been discussed in the literature [11]: Rayleigh and Langevin. The former is defined as the force acting on the wall of a closed vessel which contains the propagating medium. The latter is defined as the force exerted on a object completely surrounded by the propagating medium. The present work will focus on the Langevin radiation force which covers all vibro-acoustography applications. We shall use Eulerian coordinates, which are fixed in space. Another reference frame adopted in acoustics is the Lagrangian coordinates. This coordinate system is associated to each portion of the propagating medium.

Usually, the calculation of acoustic radiation force on an object can be divided into two steps: determine the radiation-stress tensor in the propagating medium and solve the linear acoustic scattering problem for the object. Brillouin (as reported by Borgenis [23]) was the first to explain the acoustic radiation force in ideal fluids in terms of the acoustic radiation-stress tensor defined in Eulerian coordinates. In his approach, the radiation-stress tensor is the time average of the excess of pressure and the wave momentum flux density. Cantrell [25] derived the radiation-stress tensor for lossless isotropic elastic solids as the time average of the Cauchy stress tensor and the wave momentum radiation flux. He found that the radiation force depends on the nonlinearities of the solid and the energy density of the wave. Acoustic radiation force in lossy fluids were studied by Jiang \textit{et al.} [26]

\(^1\) The one-dimensional acoustic radiation force in fluids is also called acoustic radiation pressure.
using Lagrangian coordinates and Doinikov [27] based on the Navier-Stokes equation for viscous fluids. Considerable theoretical and experimental attention has been given to problems of radiation force on spheres, plane walls, and cylinders.

Acoustic radiation force is a nonlinear phenomenon. Hence, any theory dealing with acoustic radiation force problems should at least be formulated in second-order approximation. Beissner et al. [28] showed that forth-order approximation of radiation force caused by plane waves is only 1% more accurate than second-order approximation. Rooney [29] measured the dependency of acoustic radiation force on the nonlinear parameter, \( B/A \), in fluids. He concluded that effects of \( B/A \) are small and laid within the measurement error of the experiment.

1.3.2 Applications of acoustic radiation force

The use of acoustic radiation force can be divided according to the nature of the radiation force with respect to time dependency. Some applications are based on static radiation force. More recently, applications involving time-dependent (harmonic or pulsed) radiation force have been developed. Acoustic radiation force has been used and studied in different applications as follows:

- Measure the power output of transducers in medical ultrasound machines by using the radiation force exerted by the ultrasound beam on an absorbing or reflecting target in water [30].
- Ultrasound radiometer which uses radiation force on a small sphere to measure ultrasound intensity [31].
- Oscillation of gas bubbles in liquids caused by radiation force [32].
- Measure the ultrasound power of transducers through the dynamic acoustic radiation force on a disk [33] or on a shaped-wedge vane [34].
- Measure ultrasound absorption coefficient in liquids by using radiation force of modulated sound waves [35].
- Image viscoelastic properties using radiation force produced by focused ultrasound pulses [7].
- Ultrasound vibro-acoustography.

In the first three applications, the radiation force is static and it is produced by a monochromatic sound wave, which can be assumed as a plane traveling wave or a standing wave depending on the application. The remaining applications use the dynamic acoustic radiation force.
1.3.3 Acoustic radiation force on objects

Plane walls

Consider a sound plane wave normally striking a planar perfect absorbing (or reflector) target in an ideal fluid. The radiation force exerted on the target by the incident wave is given by

\[ f = \frac{\gamma W}{c_0}, \]

where \( c_0 \) is the speed of sound in the medium and \( \gamma = 1 \) for a perfect absorber and perfect reflector \( \gamma = 2 \). The speed of sound in water is 1500 m/s, hence the radiation force on absorbing target per Watts is about \( 7 \times 10^{-4} \) N. Borginis [36] studied the radiation force caused by oblique incidence of a collimated plane wave. He concluded that the radiation force on a wedge-shaped vane (with 90°) is independent of the coefficient of reflection at the boundary between the vane and the propagating medium.

Sphere

In a pioneering work, King [37] calculated the radiation force caused by standing and traveling plane waves impinging on a rigid sphere in a compressible fluid. In this derivation thermal and viscous effects were neglected. For a small sphere (by small we mean a sphere whose diameter is much smaller than the sound wavelength) in a standing wave field, he found that the direction of the radiation force depends on the density ratio of the sphere and the surrounding medium. Heavier spheres are attracted to velocity antinodes and lighter spheres to velocity nodes. Furthermore, the acoustic radiation force exerted by a plane traveling waves are much smaller than that exerted by a standing waves of the same amplitude. Measurements of the acoustic radiation force due to a standing wave on a sphere in the air [38] are in good agreement with King’s theory. The extension of King’s result to include the sphere compressibility was presented by Yosioka et al. [39]. Crum [40] measured the acoustic radiation force caused by a stationary field necessary to trap a liquid bubble in an immiscible liquid medium at different positions. The results are in good agreement with the theory of Yosioka. Gor’kov [41] calculated the radiation force on a small compressible sphere as the gradient of the mean square fluctuation of the pressure and the velocity at the point where the sphere is located. Only the monopole and dipole terms in the scattered wave field were considered. Using King’s theory, Nyborg [42] obtained a simple formula for radiation force on small rigid sphere similar to Gor’kov’s result. The influence of the elasticity of a sphere in the radiation force has been investigated theoret-

\(^2\)A body which does not suffer any deformation even if external forces are applied is called rigid body. The speed of sound in a rigid body is infinite.
ically and experimentally by Hasegawa et al. [43, 44, 45, 46]. Chen et al. [47] presented a unified method to calculate the radiation force generated by focused axisymmetric wave fields on compressive, rigid, and solid spheres. Effects of bulk viscosity of a sphere on the acoustic radiation force were studied by Löffsted et al. [48]. The radiation force exerted by an axisymmetric sound wave on a rigid sphere in a viscous fluid was theoretically analyzed by Doinikov [27] and more recently by Danilov et al. [49]. In all cited works above, the acoustic radiation force is produced by monochromatic sound waves. Consequently, the time averaged radiation force is static. Chen et al. [19] calculated the dynamic radiation force exerted by a modulated ultrasound plane wave on a solid elastic sphere by extending the theory of Hasegawa [43] for two ultrasound plane waves at slightly different frequencies. Using a dynamic radiation force due to modulated sound waves, Marston [50] showed that a drop can be excited, deformed, and levitated in an immiscible fluid.

### Cylinders

The acoustic radiation force exerted on a freely suspended solid cylinder by a plane traveling sound field was calculated by Hasegawa [51]. Wu et al. [52] studied theoretically and experimentally the acoustic radiation force exerted by a standing sound wave on a long rigid cylinder.

### Arbitrary shaped bodies

The problem of the radiation force exerted by an arbitrary sound wave on an arbitrary shaped body is more complex than the aforementioned cases here. The problem complexity comes from how to solve the scattering problem of an arbitrary incident wave hitting an object with complex geometry. If the incident wave is a plane or axisymmetric some difficulties may be overcome. By considering a plane wave, Westervelt [53, 54] derived an expression for the acoustic radiation force stemming from Brillouin radiation-stress. This radiation force formula depends on the energy density of the incident wave multiplied by the sum of the scattering and absorption cross-sections of the target. Westervelt’s formula of radiation force has been used to model beamforming in vibro-acoustography systems.

### 1.4 Contributions of this work

The main result of this dissertation is to propose a complete model of image formation in vibro-acoustography system based on the three-dimensional radiation force vector. The contributions of this work includes:

1. Formulate and solve the problem of dynamic acoustic radiation force exerted on a point-target by any modulated sound beam in a lossless fluid. From the best of our knowledge this problem
had not been treated previously.

2. Establish an experimental method to measure the axial and transverse components of the dynamic radiation force exerted on cylinders using a laser vibrometer [55]. This work is an extension of a method to calculate the acoustic radiation force exerted by focused ultrasound beam based on ray acoustics [56]. The method can be used to measure the axial and transverse components of the ultrasound radiation force on wire in order to characterize the PSF of vibro-acoustography systems.

3. Study vibro-acoustography beamforming and resolution cell dynamics for different ultrasound transducers such as confocal and sector spherical array transducer [57]. The present vibro-acoustography system is based on a confocal or a sector array transducer. Influence of the transverse component of the ultrasound radiation force had not been studied previously.

4. Study and optimize the PSF of vibro-acoustography systems based on linear array transducers through computer simulation [58, 59, 60, 61]. The simulation is based on the spatial impulse method [62] which calculates the acoustic fields radiated by a planar piston of arbitrary geometry.

1.5 Summary

In this chapter, we introduced vibro-acoustography as an imaging technique within elastography. Acoustic radiation force was discussed as the main concept behind vibro-acoustography. Several applications of vibro-acoustography were presented and briefly discussed.
Chapter 2

Background

“I’m a traveler in both time and space
to be where I have been.” (Kashmir)
Led Zeppelin.

In this chapter, we present the concepts used throughout the vibro-acoustography image formation problem. In the mathematical background, results are presented without any rigorous proof. Acoustic wave propagation in ideal fluids is discussed starting from fluid dynamics equations. The derivation of the vibro-acoustography point-spread function (PSF) based on the collimated quasi-plane wave approximation for the modulated ultrasound beam is reviewed.

2.1 Mathematical background

Physical quantities can be represented by scalars, vectors, and tensors. Usually, these quantities are real or complex functions of the position vector $\mathbf{r}$ in the $\mathbb{R}^3$-space and the time $t$.

2.1.1 Scalars, vectors and tensors

We present some concepts and definitions of scalars, vectors and tensors. We define these mathematical objects as follows:

**Scalars**

A scalar is a quantity having magnitude but no direction, such as mass, length, time, and any real number.
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Vectors

A vector is a quantity having both magnitude and direction, such as displacement, velocity, and force.

Tensors

A \( n \)-th rank tensor in an \( m \)-dimensional space is a mathematical object that has \( n \) indices and \( m^n \) components and obeys certain coordinate system transformation rules. Tensor analysis has been applied in general relativity theory, differential geometry, mechanics, elasticity, fluid dynamics, electromagnetic theory, and other fields of science and engineering. Furthermore, second-rank tensors are used to describe acoustic radiation force problems. In this case, tensors are represented by \((3 \times 3)\)-matrices.

Dyads

A second-rank tensor in the three-dimensional space may be represented by a \((3 \times 3)\)-matrix. Another way to represent these tensors can be done by using dyads. Let \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \) be arbitrary vectors. A dyad of the vectors \( \mathbf{a} \) and \( \mathbf{b} \) is defined as \( D(\mathbf{a}, \mathbf{b}) \equiv \mathbf{a}\mathbf{b} \). The dot-product of a dyad and a vector is defined by

\[
\begin{align*}
\mathbf{a} \cdot \mathbf{c} & \equiv (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\
\mathbf{a}\mathbf{b} \cdot \mathbf{c} & \equiv (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.
\end{align*}
\]

The matrix representation of dyads can be obtained from (2.1) as follows

\[
\mathbf{ab} = (\mathbf{ba})^T = a_m b_n, \ m,n = 1,2,3,
\]\n
where superscript \( T \) denotes the matrix transpose operation.

2.1.2 Coordinate systems

The choice of which coordinate system should be used to describe a physical problem depends on the symmetry or constraints of the problem. A Cartesian coordinate system in which \( \mathbf{r} = (x,y,z) \) offers the unique advantage that all three unit vectors, \( \mathbf{e}_x \), \( \mathbf{e}_y \), and \( \mathbf{e}_z \), have constant direction in space. Sometimes we will refer to the Cartesian coordinates by \((x_1,x_2,x_3)\). Another coordinate system extensively used in this work is the spherical coordinate system. This system is defined by the
following variables:
\[
\begin{align*}
    r &= \sqrt{x^2 + y^2 + z^2}, \\
    \theta &= \arccos \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right), \\
    \phi &= \arctan \frac{y}{x}.
\end{align*}
\] (2.3)

In spherical coordinates, the unit vectors are given in terms of the Cartesian unit vector as follows
\[
\begin{align*}
    e_r &= \sin \theta \cos \phi e_x + \sin \theta \sin \phi e_y + \cos \theta e_z, \\
    e_\theta &= \cos \theta \cos \phi e_x + \cos \theta \sin \phi e_y - \sin \theta e_z, \\
    e_\phi &= -\sin \phi e_x + \cos \phi e_y.
\end{align*}
\] (2.4)

The position vector can be written as \( \mathbf{r} = r e_r \).

2.1.3 Differential operations

We shall be dealing in this work with real or complex functions of space and time. Functions will be assumed single-value, that is the value is uniquely determined by the function arguments. All differential operations will be taken on single-value functions.

**Gradient**

Let \( \xi(\mathbf{r}) \) be a scalar function differentiable at each point \( \mathbf{r} = (x,y,z) \) in a certain region in space. The gradient of \( \xi(\mathbf{r}) \) in Cartesian coordinate is defined by
\[
\nabla \xi = \frac{\partial \xi}{\partial x} e_x + \frac{\partial \xi}{\partial y} e_y + \frac{\partial \xi}{\partial z} e_z.
\] (2.5)

The gradient of \( \xi(\mathbf{r}) \) in spherical coordinates is given by
\[
\nabla \xi = \frac{\partial \xi}{\partial r} e_r + \frac{1}{r} \frac{\partial \xi}{\partial \theta} e_\theta + \frac{1}{r \sin \theta} \frac{\partial \xi}{\partial \phi} e_\phi.
\] (2.6)

**Divergence**

Let \( \mathbf{v}(\mathbf{r}) = v_1 e_x + v_2 e_y + v_3 e_z \) be a vector where \( v_1, v_2, \) and \( v_3 \) are differentiable functions.

The divergence of \( \mathbf{v}(\mathbf{r}) \) in Cartesian coordinates is defined by
\[
\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.
\] (2.7)

The definition of divergence of vectors can be expanded to matrices. Let \( \mathbf{M} = \{M_{mn}\} \) be a
differentiable $(3 \times 3)$-matrix. The dot product between $M$ and $v$ is defined by

$$(v \cdot M)_n = (M \cdot v)_n = \sum_{m=1}^{3} v_m M_{mn} \quad (2.8)$$

Then the divergence of $M$ is defined as

$$(\nabla \cdot M)_n \equiv \sum_{m=1}^{3} \frac{\partial M_{mn}}{\partial x_m}, \quad (2.9)$$

where $\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$. We notice that the divergence of a matrix is a vector. It is straightforward that the gradient of the scalar function $\xi(r)$ can be written as

$$\nabla \xi = \nabla \cdot (\xi U), \quad (2.10)$$

where $U$ is the $(3 \times 3)$-unit matrix.

Formulas involving $\nabla$

Let $a$ and $b$ be differentiable vector functions, and $\xi$ be a differentiable function. A dyad formed by $a$ and $b$ is $ab$. The following identities of differential operations hold [63]

$$\nabla (a \cdot b) = (b \cdot \nabla)a + (a \cdot \nabla)b + b \times (\nabla \times a) + a \times (\nabla \times b), \quad (2.11)$$

$$\nabla \cdot (\xi b) = \nabla \xi \cdot \mathbf{b} + \xi \nabla \cdot \mathbf{b}. \quad (2.12)$$

$$\nabla \cdot ab = (\nabla \cdot a)b + (a \cdot \nabla)b, \quad (2.13)$$

where $\times$ is the vector product and $\nabla \times \mathbf{a}$ is the rotational of $\mathbf{a}$.

The divergence theorem of Gauss

This theorem states that if a volume $V$ is bounded by a closed surface $S$ and $v(r)$ is a differentiable vector function, then

$$\int_V \nabla \cdot vdV = \oint_S v \cdot n dS, \quad (2.14)$$

where $\mathbf{n}$ is the positive (pointing outward) normal vector of $S$, and $dV$ and $dS$ are the volume and the area elements. The Gauss theorem can be extended to matrices. Let $M$ be a differentiable $(3 \times 3)$-matrix, then

$$\int_V \nabla \cdot M dV = \oint_S M \cdot n dS, \quad (2.15)$$
2.1.4 The Fourier transform

Frequently in mathematical physics we encounter pairs of functions related by an integral transform. The Fourier transform is one of the most useful of possible integral transforms. Let \( f(t) \) be a function of time. The Fourier transform \( \mathcal{F} \) of the function \( f(t) \) is defined as

\[
\tilde{f}(\omega) = \mathcal{F}\{f(t)\} \equiv \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt,
\]

(2.16)

where \( j \) is the imaginary unity and \( \omega \) denotes angular frequency. The inverse Fourier transform is given by

\[
f(t) = \mathcal{F}^{-1}\{\tilde{f}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(\omega) e^{j\omega t} d\omega.
\]

(2.17)

The existence of the Fourier transform depends on whether the function \( f(t) \) satisfies the Dirichlet’s conditions (for details see Ref. [64]). A useful representation of the Dirac delta function in terms of Fourier transform is given by

\[
\delta(\omega - \omega') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-j(\omega - \omega')t} dt.
\]

(2.18)

In connection to the Fourier transform we define the convolution operation in time. Consider two functions \( f(t) \) and \( g(t) \) with the Fourier transform \( \tilde{f}(\omega) \) and \( \tilde{g}(\omega) \), respectively. The convolution in time of \( f \) and \( g \) of is defined by

\[
f \ast g \equiv \int_{-\infty}^{+\infty} f(t')g(t-t')dt'.
\]

(2.19)

The Fourier transform of a convolution is simply the product of the Fourier transform of each individual function, i.e.,

\[
\mathcal{F}\{f \ast g\} = \tilde{f}(\omega)\tilde{g}(\omega).
\]

(2.20)

In the frequency-domain, the convolution is given by

\[
\tilde{f} \ast \tilde{g} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(\omega')\tilde{g}(\omega - \omega') d\omega'.
\]

(2.21)

The inverse Fourier transform of this expression is

\[
\mathcal{F}^{-1}\{\tilde{f} \ast \tilde{g}\} = f(t)g(t).
\]

(2.22)
2.1.5 Time average

Second or higher order acoustic quantities such as energy, intensity, power, and radiation force are usually expressed as time averaged quantities. Let \( s(t) \) be a periodic function time of frequency \( \omega_0 \). The time average of \( s(t) \) is defined as

\[
\langle s \rangle \equiv \frac{1}{T} \int_{-T/2}^{T/2} s(t) dt,
\]

(2.23)

where \( T \) is a long time interval compared to the function period.

In acoustics, quantities like energy, power, and intensity are real numbers expressed as the time average of the product of acoustic fields. In general, acoustic fields such as pressure or particle velocity are represented by a complex function. It is useful to analyze the time average of products of complex functions. Consider two functions \( s_1 \) and \( s_2 \) which vary sinusoidally in time at frequency \( \omega_0 \). When \( s_1 \) and \( s_2 \) are expressed in the complex forms \( \hat{s}_1 e^{j\omega_0 t} \) and \( \hat{s}_2 e^{j\omega_0 t} \), where \( \hat{s}_1 \) and \( \hat{s}_2 \) are the complex amplitude, the time average of the product \( s_1 s_2 \) is given by

\[
\langle s_1 s_2 \rangle = \frac{1}{2} \text{Re} \{ \hat{s}_1 \hat{s}_2^* \},
\]

(2.24)

where \( \text{Re} \) means the real part of a complex quantity.

In vibro-acoustography, the dynamic radiation force is generated by a modulated ultrasound beam. The ultrasound beam has the carrier frequency \( \omega_0 \) much larger than the modulation frequency \( \Delta \omega \). The produced radiation force has a time varying component at the modulation frequency \( \Delta \omega \). To separate this component from the total radiation force, we use the short-term time average, which for a function \( s(t) \) over an interval \( T \) at the time instance \( t \) is defined as follows:

\[
\langle s \rangle_T \equiv \frac{1}{T} \int_{t-T/2}^{t+T/2} s(t') dt'.
\]

(2.25)

Consider two time modulated function \( s_1 \) and \( s_2 \) given by

\[
\begin{align*}
\hat{s}_1(t) &= \hat{A}_1 e^{j(\omega_0 + \Delta \omega/2)t} + \hat{B}_1 e^{j(\omega_0 - \Delta \omega/2)t}, \\
\hat{s}_2(t) &= \hat{A}_2 e^{j(\omega_0 + \Delta \omega/2)t} + \hat{B}_2 e^{j(\omega_0 - \Delta \omega/2)t},
\end{align*}
\]

where \( \hat{A}_1, \hat{A}_2, \hat{B}_1, \) and \( \hat{B}_2 \) are the complex amplitudes and \( \Delta \omega \ll \omega_0 \). From the definition (2.25), one
can show that the short-term time average of the product $s_1s_2$ with $\frac{2\pi}{\omega_0} \ll T \ll \frac{2\pi}{\Delta\omega/2}$ is

$$
\langle s_1s_2 \rangle_T = \frac{1}{2} \text{Re} \{ \hat{A}_1 \hat{A}_2^* \} + \frac{1}{2} \text{Re} \{ \hat{B}_1 \hat{B}_2^* \} + \text{Re} \{ (\hat{A}_1 \hat{B}_2^* + \hat{A}_2 \hat{B}_1^*) e^{i\Delta\omega t} \} + O \left( \frac{\Delta\omega}{\omega_0} \right)^2.
$$

(2.26)

Notice that other terms at the frequency around $\omega_0$ are also present with lower amplitude in (2.26). Nevertheless, the component of $\langle s_1s_2 \rangle_T$ at $\Delta\omega$ can be exactly extracted from Eq. (2.26).

## 2.2 Acoustic wave propagation in ideal fluids

Most of the acoustic phenomena analyzed in this work take place in a fluid medium. It is assumed that the fluid medium is continuous and remains continuous under the action of any external perturbation. The molecular or atomic structure of the fluid is ignored. The medium particles are in one-to-one correspondence with the points of the $\mathbb{R}^3$-space (Euclidean space).

To study sound wave propagation in an ideal fluid, we need a set of equations describing the fluid dynamics. Ideal fluids (liquids or gases) are fluids where thermal conductivity and viscosity are unimportant. Hence no energy dissipation or heat transfer between parts of the fluid or with the exterior occur. The lack of heat transfer means that the wave propagation is adiabatic throughout the medium. The fluid medium is characterized by the following acoustic fields: pressure $P$, density $\rho'$, specific entropy $s$ (entropy per unit mass), and particle velocity $v$. All fields are functions of space and time. In an initial state without sound field these quantities correspond, respectively, to the following spatially constant quantities $P = P_0$, $\rho' = \rho_0$, $s = s_0$, and $v = 0$. The quantity $P - P_0$ is the acoustic pressure. Several text books [65, 66] derive the equations of fluid dynamics for ideal fluids stemming from conservation principles for mass, momentum, entropy, and thermodynamic equilibrium. We present these equations here when no external sources of mass, force, or energy are acting on the fluid:

$$
\begin{align*}
\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho' v) &= 0 \quad \text{(equation of continuity)} \\
\rho' \left[ \frac{\partial v}{\partial t} + (v \cdot \nabla) v \right] + \nabla P &= 0 \quad \text{(Euler’s equation)} \\
\frac{\partial s}{\partial t} + v \cdot \nabla s &= 0 \quad \text{(adiabatic hypothesis)} \\
\rho' &= P(\rho', s) \quad \text{(equation of state)}.
\end{align*}
$$

(2.27)

The equation of continuity states that the mass is conserved in the fluid. The Euler’s equation represents the momentum conservation in the fluid. The adiabatic hypothesis states that the entropy of the fluid is constant. The equation of state $P = P(\rho', s)$ can be expanded in a Taylor series along the
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isentrope $s = s_0$ as follows

$$P = P_0 + \left( \frac{dP}{dp_0'} \right)_0 (p' - p_0) + \frac{1}{2!} \left( \frac{d^2P}{dp_0'^2} \right)_0 (p' - p_0)^2 + \ldots,$$  \hspace{1cm} (2.28)

where the subscript 0 indicates that the partial derivatives are evaluated at the unperturbed state $(\rho_0, s_0)$.

The equations of fluid dynamics given in (2.27) are nonlinear partial differential equations. These equations give a full description of the wave motion in the fluid for a specific initial and boundary conditions. To obtain exact solutions of (2.27) is a difficult task even for a simple case. However, considering waves with small amplitude, it is possible to obtain a linear wave equation to describe the fluid. We assumed that the pressure and the density fields are linear quantities given by

$$P(r, t) = P_0 + p(r, t)$$  \hspace{1cm} (2.29)

$$\rho_0(r, t) = \rho_0 + \rho(r, t),$$  \hspace{1cm} (2.30)

where $p$ and $\rho$ are the acoustic pressure and density, respectively. In this approximation, the equation of continuity becomes

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0.$$  \hspace{1cm} (2.31)

From the equation of state (2.28) we see that for small changes the relation between pressure and density is given by

$$p = \left( \frac{dP}{dp_0'} \right)_0 \rho = c_0^2 \rho,$$  \hspace{1cm} (2.32)

where the $c_0$ is the small signal speed of sound. Substituting this result in the continuity equation (2.31), we find

$$\frac{\partial \rho}{\partial t} + \rho_0 c_0^2 \nabla \cdot \mathbf{v} = 0.$$  \hspace{1cm} (2.33)

The second-order term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ in the Euler’s equation may be neglected. Hence, the Euler’s equation becomes

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0.$$  \hspace{1cm} (2.34)

Eqs. (2.33) and (2.34) give a complete description of the sound wave.

The velocity and pressure fields can be expressed in terms of a scalar potential function $\phi$. By using the Helmholtz’s theorem [67] and the fact that the fluid is lossless (irrotational), the velocity field is given by $\mathbf{v} = -\nabla \phi$. Substituting $\mathbf{v}$ into (2.34) we get

$$p = \rho_0 \frac{\partial \phi}{\partial t}.$$  \hspace{1cm} (2.35)
Now, substituting \( v \) and \( p \) in terms of the acoustic potential \( \phi \) into Eq. (2.33) yields

\[
\nabla^2 \phi - \frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} = 0.
\]

(2.36)

This is the linear wave equation. This equation defines a boundary value problem for which given the boundary conditions the solution is unique. By taking the Fourier transform \( \mathcal{F} \) of (2.36), we obtain the wave equation in frequency-domain (the Helmholtz’s equation) as follows

\[
\nabla^2 \tilde{\phi} + \left( \frac{\omega}{c_0} \right)^2 \tilde{\phi} = 0,
\]

(2.37)

where \( \tilde{\phi} = \mathcal{F} \{ \phi \} \). The Green’s function of the linear wave equation satisfies the following partial differential equation

\[
\nabla^2 G(\mathbf{r} | \mathbf{r}'; \omega) + \left( \frac{\omega}{c_0} \right)^2 G(\mathbf{r} | \mathbf{r}'; \omega) = -\delta(\mathbf{r} - \mathbf{r}').
\]

(2.38)

The Green’s function \( G(\mathbf{r} | \mathbf{r}'; \omega) \) can be interpreted as a monopole source placed at \( \mathbf{r}' \) radiating an outgoing spherical waves.

In this work, the wave equation will be used in the following subjects: acoustic scattering, calculation of acoustic radiation force, acoustic emission, and radiation pattern of transducers for vibro-acoustography beamforming. It is important to understand that the solutions of Eq. (2.36) are valid when the wave amplitude is small and the fluid is homogeneous. By homogeneous we mean that in the absence of sound the acoustic fields do not vary spatially. If the wave amplitude is not small, Eq. (2.36) will not give a satisfactory result. In this case, nonlinear terms in Eq. (2.28) should be considered.

### 2.3 Ultrasound vibro-acoustography

The principle behind ultrasound vibro-acoustography is based on the fact that different objects produce different sound emission when tapped by an exterior force. Most people are able to distinguish the sound emitted by a tapped metal compared to wood tapped by the same force. Of course metal and wood are quite different materials. The sound emitted by a tapped object reveals information about its mechanical properties. The emission depends also on the object geometry. Vibro-acoustography uses ultrasound radiation force to “tap” an object and produce a map of the mechanical response of the object. The ultrasound intensity is modulated to generated a dynamic localized radiation force field at the modulation frequency on the probed object.

The spatial resolution of the system is related to the volume enclosed by the radiation force
distribution at $-12$ dB from its peak. A high resolution system requires this volume to be as small as possible. The spatial resolution of a system based on a two-element confocal transducer in the transverse direction is about 700 $\mu$m at $-6$ dB [9]. Figure 2.1 shows the theoretical result of the radiation stress on the focal plane of the confocal two-element array transducer based on the quasi-plane wave approximation model for the modulated ultrasound beam [68]. We shall show that the PSF of a vibro-acoustography operating in the AM-single beam mode is a particular case of dual beam mode. However, the resulting ultrasound beam in dual beam mode is modulated only in the focal region of the transducer. While in AM-single beam mode, the ultrasound beam is modulated along the wave path.

2.3.1 Dynamic ultrasound radiation force

To produce the dynamic radiation force we consider dual beam mode. In this mode, two intersecting ultrasound beams, $a$ and $b$ focused at $r_0$, produce the dynamic radiation force on an object. Each ultrasound source is driven by a continuous-wave (CW) signal at frequencies $\omega_a = \omega_0 + \Delta\omega/2$ and $\omega_b = \omega_0 - \Delta\omega/2$, where $\omega_0$ is the angular center frequency, $\Delta\omega$ is the angular difference frequency, and $\Delta\omega \ll \omega_0$.

In the plane wave model, two collimated plane waves at frequencies $\omega_a$ and $\omega_b$ intersect each other in the propagating medium. The overlapping region defines the focal zone of the system. The direction of propagation of the waves is given by the unit vector $e_a$ and $e_b$, respectively. Let $r_d$
Figure 2.2: Illustration of the plane wave approximation model for vibro-acoustography beamforming. The small planar target is placed at $r_1$ tangent to the surface $\Omega$. The resulting ultrasound beam is focused at $r_0$. The acoustic emission by the target is detect at $r_2$. and $r_b$ be the position of the ultrasound sources. The direction of propagation of the plane waves are given by $e_a = \frac{r_0 - r_a}{|r_0 - r_a|}$ and $e_b = \frac{r_0 - r_b}{|r_0 - r_b|}$. See Figure 2.2 for geometric description. We assume that the vectors $e_a$ and $e_b$ are coplanar. In the focal zone the acoustic potential is given by

$$\phi_D(r,t) = \left(\frac{2Ic_0}{\rho_0}\right)^{1/2} \text{Re} \left\{ \frac{e^{j(\omega_a t - k_a \cdot (r - r_a))}}{\omega_a} + \frac{e^{j(\omega_b t - k_b \cdot (r - r_b))}}{\omega_b} \right\}, \quad (2.39)$$

where $I$ is the sound intensity, $r_a$ and $r_b$ are the position of the ultrasound sources, $k_a = \frac{\omega_0 + \Delta \omega}{c_0} e_a$, and $k_b = \frac{\omega_0 - \Delta \omega}{c_0} e_b$.

The radiation force caused by a sound wave on the area element $dS$ of an object is given by [69]

$$dF = \langle -L n + \rho_0 (v \cdot n) v \rangle dS, \quad (2.40)$$

where $L = \frac{\rho_0 (v \cdot v)}{2} - \frac{L^2}{2\rho_0 a^2}$ is the Lagrangian density of the wave, and $n$ is the normal vector of $dS$. The total radiation force on the object can be obtained by integrating (2.40) on the object surface. Westervelt [53,54] showed that the normal component of the radiation force caused by an infinite
extended or collimated plane wave striking an object of arbitrary shape is given by

\[ f_n = d_r \langle E \rangle, \quad (2.41) \]

where \( d_r \) is a function of the scattering and absorption cross-sections by the object and \( \langle E \rangle \) is the time averaged energy density of the sound wave.

To simplify the analysis in this problem, consider a planar perfect absorber in the intersecting region of the ultrasound beams. The ultrasound radiation force on the infinitesimal area \( dS \) of the perfect absorber can be obtained by substituting (2.39) into (2.40) and taking the short-term time average defined in (2.25). Accordingly,

\[ df = \frac{2I_c}{c_0} \left[ (1 - e_a \cdot e_b) \mathbf{n} + (e_a \cdot \mathbf{n}) e_b + (e_b \cdot \mathbf{n}) e_a \right] \cos \left[ \Delta \omega t - k_a \cdot (r - r_a) + k_b \cdot (r - r_b) \right]. \quad (2.42) \]

If the area element \( dS \) is perpendicular to the radiation force we have the condition

\[ (e_a \cdot \mathbf{n}) (e_b \cdot \mathbf{m}) + (e_b \cdot \mathbf{n}) (e_a \cdot \mathbf{m}) = 0, \quad (2.43) \]

where \( \mathbf{m} \) is a unit vector perpendicular to \( \mathbf{n} \). Consider \( \theta_a \) and \( \theta_b \) defined by \( \cos \theta_a = e_a \cdot \mathbf{n} \) and \( \cos \theta_b = e_b \cdot \mathbf{n} \). From Eq. (2.42) we have \( \theta_a \) = \( \theta_b \). Hence the normal vector to \( dS \) is \( \mathbf{n} = \frac{e_a + e_b}{|e_a + e_b|} \). In other words,

\[ \mathbf{n} = \frac{(|r_0 - r_a| + |r_0 - r_b|) \mathbf{r}_0 - (|r_0 - r_a| r_a + |r_0 - r_b| r_b)}{|r_0 - r_a| |r_0 - r_b|}. \quad (2.44) \]

The vector \( \mathbf{n} \) defines the surface \( \Omega \) that is perpendicular to dynamic component of the radiation force. In fact, any surface whose normal vector is \( \mathbf{n} \) passing through the focal region of the transducer is perpendicular to the dynamic component of the radiation force. If the sources are symmetric and \( |r_0| \gg |r_a| = |r_b| \), or the sources are located in the same point of the space the vector \( \mathbf{n} \) becomes

\[ \mathbf{n} = \frac{r_0}{|r_0|}. \quad (2.45) \]

In this case, the vector \( \mathbf{n} \) defines a cylindrical concave surface.

Consider two collimated ultrasound waves whose spatial amplitudes vary in the transverse direction of the wave propagation. The transverse variation in each wave is assumed to vary slower compared to that in the direction of the wave propagation. Hence, we can consider these waves as collimated quasi-plane waves. The acoustic potential in the medium is given by

\[ \phi_D(r, t) = \left( \frac{2I_c}{\rho_0} \right)^{1/2} \Re \left\{ \frac{1}{\omega_a} \hat{\phi}_a (r_0) e^{i\omega_a t} + \frac{1}{\omega_b} \hat{\phi}_b (r_0) e^{i\omega_b t} \right\}, \quad (2.46) \]
where \( D \) denotes dual beam mode, \( I \) is the ultrasound average intensity and \( \hat{\phi}_a (r|\mathbf{r}_0) \) and \( \hat{\phi}_b (r|\mathbf{r}_0) \) are the complex dimensionless amplitude of the ultrasound beams. The notation given in the amplitude functions of the ultrasound beams is to emphasize that the beams are focused at \( \mathbf{r}_0 \). This will be useful to define the system point-spread function. Consider an AM-single ultrasound beam described by the acoustic potential

\[
\phi_S(r, t) = \left( \frac{2Ic_0}{\rho_0\omega_0^2} \right)^{1/2} \operatorname{Re} \left\{ \hat{\phi}_0 (r|\mathbf{r}_0) e^{i\omega_0 t} \cos \left( \frac{\Delta\omega}{2} t \right) \right\},
\]

(2.47)

where \( S \) stands for AM-single beam and \( \hat{\phi}_0 \) is the dimensionless complex amplitude of the AM beam. Using the identity \( \cos\alpha \cos\beta = \frac{1}{2} \left[ \cos(\alpha - \beta) + \cos(\alpha + \beta) \right] \), where \( \alpha \) and \( \beta \) are two arbitrary angles, one can show that

\[
\phi_S(r, t) = \frac{1}{2} \left( \frac{2Ic_0}{\rho_0\omega_0^2} \right)^{1/2} \operatorname{Re} \left\{ \hat{\phi}_0 (r|\mathbf{r}_0) \left[ e^{i\omega_0 t} + e^{i\omega_0 t} \right] \right\}.
\]

(2.48)

We see from Eqs. (2.47) and (2.48) that single beam mode is a particular case of dual beam mode in which \( \hat{\phi}_a = \frac{\omega_0}{2\Delta\omega} \hat{\phi}_0 \) and \( \hat{\phi}_b = \frac{\omega_0}{-2\Delta\omega} \hat{\phi}_0 \). We continue our analysis based on dual beam mode.

The instantaneous ultrasound energy density of the collimated quasi-plane wave is given by

\[
E = \frac{\rho_0}{c_0^2} \left( \frac{\partial \phi_D}{\partial t} \right)^2.
\]

(2.49)

The energy density produced by the incident field in (2.46) has slow variations in time with frequency \( \Delta\omega \). To separate this slow variation in time we use the short-term time average defined in 2.25. The energy density is calculated by substituting (2.46) into (2.49). The short-term time average of the energy density, with \( T \) satisfying the condition \( \frac{\pi}{\Delta\omega} \ll T \ll \frac{4\pi}{\Delta\omega} \), is given by

\[
\langle E \rangle_T = \frac{I}{c_0} \left\{ \left| \hat{\phi}_a (r|\mathbf{r}_0) \right|^2 + \left| \hat{\phi}_b (r|\mathbf{r}_0) \right|^2 + 2\operatorname{Re} \left[ \hat{\phi}_a (r|\mathbf{r}_0) \hat{\phi}_b^* (r|\mathbf{r}_0) e^{i\lambda\omega t} \right] \right\}.
\]

(2.50)

A linear imaging system is characterized by its point-spread function (PSF), which is defined as the response of the system to a point target. In ultrasound imaging, the point-target at \( r_1 \) can be represented by an infinitesimal volume \( dV \) times the 3D-delta function \( \delta^3(r - r_1) \). The vibro-acoustography PSF depends on the generated dynamic component of the radiation force on the point-target by the ultrasound beam. The radiation force produced on an object is a vector quantity in three-dimensional space. To simplify the system analysis, the point target is represented by an infinitesimal planar object with area \( dS \) with zero-thickness located at \( r_1 \). The orientation of \( dS \) is such that it is always perpendicular to the dynamic component of the radiation force. Thus, the radiation force can be treated as a scalar quantity.

\footnote{The transverse gradient of a collimated quasi-plane wave can be neglected.}
In the quasi-plane wave approximation model, it is possible to show that there is a surface \( \Omega \), which is always perpendicular to the dynamic component of the radiation force as described previously. The origin of the coordinate system coincides with the geometric center of the transducer aperture. The small object (point-target) is placed at \( r_1 \) tangent to the surface \( \Omega \). The object is free to vibrate only in the direction of the dynamic component of the radiation force. These assumptions allow us to formulate the problem using only the normal component of the force, which is obtained by substituting the dynamic part of (2.50) in (2.41). The result is

\[
L_n(t) = \frac{2Id_r}{c_0} \Re \left\{ \hat{\phi}_a(r_1 | r_0) \hat{\phi}_b^*(r_1 | r_0) e^{i\Delta\omega t} \right\}. 
\]

### 2.3.2 Acoustic emission and point-spread function

The dynamic radiation force vibrates the object target at the difference frequency \( \Delta\omega \). In response, the object emits an acoustic field (acoustic emission). The acoustic emission is proportional to the dynamic component of the radiation force and related to the size, shape, and mechanical properties of the object. The complex amplitude of the acoustic emission detected at point \( r_2 \) by an object centered at \( r_1 \) can be expressed as [68]:

\[
p(r_2; r_1 | r_0) = \frac{2Id_r}{c_0} \hat{\phi}_a(r_1 | r_0) \hat{\phi}_b^*(r_1 | r_0) Q(r_1) G(r_2; r_1),
\]

where \( r_0 \) is the focus point of the transducer, \( G(r_2; r_1) \) is the medium transfer function and \( Q(r_1) \) is the acoustic outflow by the object per unit of force. The acoustic outflow is the total volume of the medium in front of the object surface that is displaced per unit time due to the object vibration. The function \( Q(r_1) \) represents the dynamic characteristics of the object at the frequency \( \Delta\omega \). These characteristics depend on the geometry, mechanical parameters, and the boundary conditions of the object.

Vibro-acoustography image synthesis is based on the acoustic emission in Eq. (2.52). To obtain the vibro-acoustography PSF, the point-target is considered as an infinitesimal planar object (representing a volume \( dV \)) and a function representing the object to be imaged should be defined from (2.52). By assuming that the function \( G(r_2; r_1) \) is independent of the object, the characteristics of the object in Eq. (2.52) can be represented by \( d_r \) and \( Q(r_1) \). For an arbitrary object, these quantities may vary spatially. Therefore, the function representing the object is defined as

\[
\xi(r) \equiv d_r(r)Q(r). 
\]

This allows us to define a unit point-target as \( \xi(r_1) = \delta^3(r - r_1) \). To calculate the PSF, we assume
that for a fixed focal point \( r_0 \), the object can vary its position in the neighborhood of \( r_0 \). This neighborhood is assumed to be small enough such that \( G(r_2, r_1) \) remains unchanged. Thus, the normalized PSF of a vibro-acoustography coherent imaging system is defined in terms of the amplitude of the expression (2.52) as follows

\[
\hat{h}_{psf}(r|r_0) = \frac{\hat{\phi}_a(r|r_0) \hat{\phi}_b^*(r_0|r)}{\hat{\phi}_a^*(r_0|r_0) \hat{\phi}_b(r_0|r_0)},
\]

where we dropped the subindex in \( r_1 \). The notation used above emphasizes that the PSF depends on \( r_0 \) (focal point), hence the system is spatially shift-variant. Inasmuch as the point-target can be placed anywhere in space, the PSF defined in (2.54) is a three-dimensional function of the space coordinates. The expression (2.54) is valid for transducers with any geometrical shape. The spatial resolution cell of the system is defined as volume enclosed by the system PSF at \(-12\) dB, which corresponds to \( 25\% \) of its the PSF peak value. This definition is arbitrary. In Refs. [9, 68], the resolution cell was considered at \( 50\% \) \((-6\) dB) of the PSF peak. Notice from Eq. (2.54) that the resolution cell depends on the relative position of the sources \( a \) and \( b \). For a vibro-acoustography system operating in AM-single beam mode the PSF becomes

\[
\hat{r}_{psf}(r|r_0) = \left[ \frac{\hat{\phi}_0(r|r_0)}{\hat{\phi}_0^*(r_0|r_0)} \right]^2.
\]

In general, a vibro-acoustography imaging system is shift-variant. However, we may consider the system to be shift-invariant within a small neighborhood of the focal point. Hence, the image of a given object \( \xi(r) \) in the neighborhood of \( r_0 \) is expressed by the spatial convolution of the function \( \xi(r) \) and the system PSF, i.e.,

\[
\chi(r) = \hat{h}_{psf}(r|r_0) \ast \xi(r).
\]

In a system based on linear arrays we shall show that the PSF suffers distortions as the ultrasound incident beam is steered. Indeed, the system may be considered spatially shift-invariant only within a region in space where the distortion effects are minimum. Therefore, Eq. (2.56) is strictly valid only within this region. If the object extends beyond the neighborhood of \( r_0 \), then we have to use a different PSF for each region, i.e., the convolution in (2.56) is calculated in a piecewise shift-invariant manner.

In conventional ultrasound terminology, two image formats are commonly used. Assuming that the ultrasound beam propagates in the \( z \)-direction, B-scan images correspond to a scanning plane parallel to the \( xz \)-plane, and C-scan images correspond to a scanning plane parallel to the \( xy \)-plane (azimuth plane). Vibro-acoustography can produce B- and C-scan images by steering and focusing the ultrasound beams \( a \) and \( b \) at different points on the desired scanning plane.

The theory of vibro-acoustography image formation presented in this section is the generalized form of the theory presented in [68] for the particular case of a confocal transducer.
2.3.3 Vibro-acoustography system description

The experimental setup used to produce vibro-acoustographic images is described in Figure 2.3. A two-element confocal ultrasound transducer is used to produce the modulated ultrasound beam. Each element is driven at 3 MHz and 3 MHz + $\Delta f$ by a stable radio-frequency (RF) synthesizer (HP-33120A, Hewlett-Packard). Geometrically the two ultrasound beams meet in a small region at the transducer focal zone. The acoustic emission caused by the object vibration is measured through a hydrophone (ITC-6080C, International Transducer Corp.). The probed object, the transducer, and the hydrophone are immersed in a tank filled with degassed water. The detected signal is filtered at $\Delta f$ and amplified by a lock-in amplifier (7265-DSP Lock-in Amplifier, Perkin Elmer Instruments). The filtered signal is digitized by a 12 bits/sample digitizer (HP-E1429A, Hewlett-Packard) at the rate higher than the Nyquist rate required for the particular $\Delta f$ used. The digitized data is recorded and displayed in a computer.

2.4 Rationale

We have discussed a model for image formation for vibro-acoustography system in the previous section. However, the imaging formation process is still not entirely understood. So far in vibro-acoustography theory, the radiation force is obtained using the collimated quasi-plane wave approximation for the intersecting ultrasound beams [68, 59]. Moreover, the point-target is assumed to be a small planar object. The ultrasound waves is constrained to strike the object normally. In this approximation, the acoustic radiation force is reduced to a scalar quantity. Hence, only the component of the radiation force in the direction of the ultrasound wave propagation is considered. Chen et al. [70]
studied the beamforming in vibro-acoustography based on the quasi-plane wave approximation model for ultrasound beams produced by transducers operating in dual confocal and x-focal\(^2\) beams, and AM-single beam. In this study, the theory is validated by measuring only the axial component of the dynamic radiation force on a small sphere. For transducer with fixed focal distance, the scalar radiation force gives a satisfactory description of the system. However, for linear array transducers, whose the focal distance can be vary and depth and be steered, the scalar radiation force theory imposes few complicated assumptions for the target and the ultrasound waves.

In fact, the acoustic radiation force is a three-dimensional vector quantity. The goal of this work is to present a theory of the dynamic radiation force exerted by any modulated ultrasound wave on a point-target stemming from the Brillouin radiation-stress tensor. The radiation force is calculated by using acoustic scattering theory. The theory is applied to systems based on spherical confocal and sector transducers, and linear array transducers.

### 2.5 Summary

We have presented key concepts necessary to the development of this work. Special attention was given to the mathematical background. Wave propagation in ideal fluids was described based on the linear wave equation which was derived stemming from the fluid dynamics equations. Vibro-acoustography theory based on quasi-plane wave model for the radiation force was presented.

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\(^2\)In this mode, the two ultrasound beams cross each other spatially resembling the letter ’x’.
Chapter 3

Ultrasound radiation force

“A lack of information cannot be remedied by any mathematical trickery.”
Lanczos.

Vibro-acoustography is strictly related to ultrasound radiation force. In this chapter, we describe acoustic radiation force stemming from the fluid dynamics equations (2.27). The connection between ultrasound radiation force and acoustic scattering is discussed. The scattering theory is applied to calculate the radiation force caused by a modulated ultrasound beam hitting a point-target. This problem is specifically important to characterize a vibro-acoustography imaging system. The notation used in this chapter is the same used in Sec. 2.2.

3.1 Ultrasound radiation force in an ideal fluid

The approach to calculate ultrasound radiation force is based on general equations of fluid dynamics for ideal fluids. Effects of viscosity and thermal conduction are not considered. Thus no dissipation of energy occurs during the wave propagation. Effects of the gravitational field are neglected.

We want to obtain the conservation equation for the acoustic radiation-stress (Brillouin stress tensor). Beforehand, let us write two important relation for the velocity field $v$. Using Eqs. (2.1) and (2.13) one can derive the following expressions

\[
(v \cdot \nabla)v = \nabla \cdot vv - (\nabla \cdot v)v, \tag{3.1}
\]

\[
\rho' \nabla \cdot vv = \nabla \cdot (\rho'vv) - (\nabla \rho') \cdot v. \tag{3.2}
\]
By combining the density conservation equation in (2.27), (3.1), and (3.2) in the momentum equation given by (2.27), we have
\[ \frac{\partial (\rho' \mathbf{v})}{\partial t} + \nabla \cdot (\rho' \mathbf{vv}) + \nabla P = 0, \] \hspace{1cm} (3.3)
where \( \rho' \mathbf{v} \) is the momentum density and \( \rho' \mathbf{vv} \) is the momentum flux tensor. It does not make any difference if \( P \) is replaced by \( P - P_0 \) in Eq. (3.3). Furthermore, to keep Eq. (3.3) in second-order, \( \rho' \) is replaced by \( \rho_0 \) in the second term of LHS of this equation. Thus using expression (2.10) in Eq. (3.3) yields
\[ \frac{\partial (\rho' \mathbf{v})}{\partial t} + \nabla \cdot [(P - P_0)\mathbf{U} + \rho_0 \mathbf{vv}] = 0. \] \hspace{1cm} (3.4)
The expansion of \( P - P_0 \) in second-order is given by [11]
\[ P - P_0 = p + \frac{p^2}{2\rho_0 c_0^2} - \frac{\rho_0 (\mathbf{v} \cdot \mathbf{v})}{2}, \] \hspace{1cm} (3.5)
where \( p \) is the linear acoustic pressure. By substituting (3.5) into (3.4) and using \( \nabla p = -\rho_0 \frac{\partial \mathbf{v}}{\partial t} \) we have
\[ \frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \sigma + \nabla p = 0, \] \hspace{1cm} (3.6)
where
\[ \sigma = -\mathcal{L} \mathbf{U} + \rho_0 \mathbf{vv} \] \hspace{1cm} (3.7)
is the second-order radiation stress tensor. The quantity \( \mathcal{L} \) in Eq. (3.7) is the Lagrangian density of the sound field given by
\[ \mathcal{L} = \frac{\rho_0 (\mathbf{v} \cdot \mathbf{v})}{2} - \frac{p^2}{2\rho_0 c_0^2}. \] \hspace{1cm} (3.8)
Now we take the time average (2.23) of Eq. (3.6). The result is
\[ \nabla \cdot \langle \sigma \rangle = 0. \] \hspace{1cm} (3.9)
Notice that \( \left\langle \frac{\partial (\rho \mathbf{v})}{\partial t} \right\rangle = 0 \) and \( \langle p \rangle = 0 \). It should be remarked that the time average taken in (3.9) depends on the scale of observation of the radiation force phenomenon. Most observation done experimentally are interested to measure the static radiation force produced by monochromatic sound waves. In that case, the time average can be defined over a wave cycle. However, vibro-acoustography is based upon dynamic (time-varying) radiation force with frequencies up to 100 kHz. To separate the dynamic component of the radiation force, we shall use the short-term time average as defined in (2.25).

The radiation-stress averaged in time is a zero divergence quantity in space. Two consequences of this result should be pointed out: if there is no target in the wave path, there is no radiation...
force; if a target is present in the wave path, the radiation force can be obtained by integrating \( \langle \sigma \rangle \) over any surface enclosing the target. Therefore, the radiation force produced on a target is

\[
f = - \int_S \langle \sigma \rangle \cdot \mathbf{n} dS = \int_S \left( L_{ii} - \rho_0 (\mathbf{v}_i \cdot \mathbf{n}) \mathbf{v}_i \right) dS,
\]

(3.10)

where \( S \) is a surface enclosing the target and \( \mathbf{n} \) is the unit normal vector of \( S \) pointing inward.

We should analyze the contribution of the incident and scattered waves to the radiation force. The sound scattering by an object is followed by the arising of the acoustic radiation force upon the object. This can be understood in terms of the momentum transfer process. An incident wave carries a momentum flux, so does the scattered wave. The difference between the momentum flux of the incident and scattered waves contributes to the radiation force exerted on the object. Hence, the radiation force on a object can be obtained by solving the scattering problem for the object. The total particle velocity and pressure in the medium can be written as a sum of the incident (subscript \( i \)) and scattered (subscript \( s \)) fields as follows

\[
\mathbf{v} = \mathbf{v}_i + \mathbf{v}_s,
\]

(3.11)

\[
p = p_i + p_s.
\]

(3.12)

Assume that the surface \( S \) described in Eq. (3.10) is very far away from the target (control sphere). In terms of the incident and scattered fields, the radiation force can be expanded as

\[
f(t) = \int_S \left( \left( L_{ii} + L_{is} + L_{ss} \right) \mathbf{n} 
- \rho_0 \left[ (\mathbf{v}_i \cdot \mathbf{n}) \mathbf{v}_i + (\mathbf{v}_s \cdot \mathbf{n}) \mathbf{v}_s + (\mathbf{v}_i \cdot \mathbf{n}) \mathbf{v}_i + (\mathbf{v}_s \cdot \mathbf{n}) \mathbf{v}_s \right] \right) dS.
\]

(3.13)

where

\[
\begin{align*}
L_{ii} &= \frac{\rho_0 (v_i \cdot v_i)}{2} - \frac{p_i^2}{2 \rho_0 c_0^2}, \\
L_{is} &= \rho_0 (v_i \cdot v_s) - \frac{\rho_0 p_s}{\rho_0 c_0^2}, \\
L_{ss} &= \frac{\rho_0 (v_s \cdot v_s)}{2} - \frac{p_s^2}{2 \rho_0 c_0^2}
\end{align*}
\]

(3.14)

are the Lagrangian densities terms of the incident, interference, and scattered waves, respectively. If there is no target on the wave path, no radiation force arises. Thus, the contribution of the incident wave yields

\[
\int_S \left( L_{ii} \mathbf{n} - \rho_0 (\mathbf{v}_i \cdot \mathbf{n}) \mathbf{v}_i \right) dS = 0.
\]

(3.15)

Westervelt [54] stated that the Lagrangian of the scattered wave should be zero when the observation point is very far way from the scatterer. He argued that the scattered waves in this region are very
close to plane waves. Thus,
\[ \int_S \langle L_{ss} \rangle \mathbf{n} dS = 0. \]  
(3.16)

Therefore, the contributions to the radiation force come from the interference of the incident and scattered waves and the momentum transferred to the object. Accordingly,
\[ \mathbf{f} = \mathbf{f}_{\text{ir}} + \mathbf{f}_{\text{ss}} \]  
(3.17)

where
\[ \mathbf{f}_{\text{ir}} = \int_S \langle L_{ir} \rangle \mathbf{n} - \rho_0 \left[ (\mathbf{v}_s \cdot \mathbf{n}) \mathbf{v}_i + (\mathbf{v}_i \cdot \mathbf{n}) \mathbf{v}_s \right] dS, \]  
(3.18)
\[ \mathbf{f}_{\text{ss}} = -\rho_0 \int_S \langle (\mathbf{v}_s \cdot \mathbf{n}) \mathbf{v}_s \rangle dS. \]  
(3.19)

### 3.2 Acoustic scattering by a point-target

Scattering problems concern how an incident wave is affected by an object. In the *direct scattering theory*, the scattered wave is determined from a knowledge of the incident wave and the equation governing the wave motion. If the object is much larger than the wavelength reflection and refraction of the sound wave occur. If the object is much smaller than the wavelength, the scattered sound wave propagates in all directions in the medium. Exact analytic solutions of the direct scattering problem in fluids are well known for incident plane waves scattered by solid objects with simple geometry such as spheres and cylinders [71, 72]. For objects with arbitrary shape, analytical solutions are not available. This is mostly because arbitrary shaped objects may not have spatial symmetry. However, scattering problems in ideal fluids for objects with any shape can be formulated through the
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Figure 3.2: Scattering by an inhomogeneous region characterized by a density $\rho_V(r)$ and compressibility $\kappa_V(r)$.

Lippmann-Schwinger equation [73]. This equation can be solved numerically using the fast Fourier transform [74] or spectral methods [75]. Figure 3.1 illustrates the scattering of an arbitrary shaped object.

A sound wave can be scattered also by an inhomogeneous region [76] in which its mechanical properties, like density and compressibility, differ from their values in the rest of the medium. Models of wave scattering in soft tissue by inhomogeneities have been applied to describe ultrasound imaging systems [77, 78]. Soft tissues are formed by different components, such as muscle and fat, with different density and compressibility. If the amplitude variation is small, the scattered wave can be obtained from the interaction of the incident wave with the inhomogeneities. This case corresponds to weak scattering which can be solved through the so-called Born approximation. With larger amplitude variation, multiple scattering can occur producing reverberation in the medium.

Consider an inhomogeneous region with volume $V$ inside a homogeneous ideal fluid of density $\rho_0$, sound speed $c_0$ and compressibility $\kappa_0 = \frac{1}{c_0^2 \rho_0}$. The fluid is considered to be of infinite extent. Without sound waves the region $V$ has density $\rho_V(r)$, speed of sound $c_V(r)$, and compressibility $\kappa_V(r) = \frac{1}{c_V^2 \rho_V}$ (see Figure 3.2). Because the density and the speed of sound of the unperturbed fluid does vary spatially, Eq. (2.33) inside the inhomogeneous region becomes

$$\frac{\partial p}{\partial t} + c_V^2(r) \left[ \frac{d\rho_V(r)}{dt} + \rho_V(r) \nabla \cdot v \right] = 0.$$  \hspace{1cm} (3.20)
By using the relation \( \frac{\partial v}{\partial t} = -\nabla p / \rho_v(r) \) and \( \frac{\partial \rho_v}{\partial t} = 0 \) in Eq. (3.20), we get
\[
\mathbf{\nabla} \cdot \left[ \frac{\nabla p}{\rho_v(r)} \right] - \kappa_v(r) \frac{\partial^2 p}{\partial t^2} = 0. \tag{3.21}
\]

Based on Eq. (3.21), the wave equation for the entire region can be written as
\[
\nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} = -\left( U_\kappa + U_\rho \right) p, \tag{3.22}
\]
where
\[
U_\kappa = -\frac{\kappa_v(r)}{c_0^2} \frac{\partial^2}{\partial t^2} \quad \text{and} \quad U_\rho = -\mathbf{\nabla} \cdot [\gamma_\rho(r) \mathbf{\nabla}] \quad \text{with}
\]
\[
\gamma_\kappa(r) = \begin{cases} \frac{\kappa_v}{\kappa_0} - 1, & r \in V \\ 0, & r \notin V \end{cases}, \quad \gamma_\rho(r) = \begin{cases} 1 - \frac{\rho_v}{\rho_0}, & r \in V \\ 0, & r \notin V \end{cases}. \tag{3.23}
\]

Eq. (3.22) is the inhomogeneous wave equation. The terms \( U_\kappa \) and \( U_\rho \) are the potential functions responsible for the scattering by the inhomogeneities. In fact, in the time-domain these functions are differential operators in time and space. Notice that the scattering potential has a contribution from the compressibility and the density of the inhomogeneous region. Scattering produced by variations in compressibility has a monopole radiation pattern. Scattering due to variations in density has a dipole radiation pattern.

It is convenient to analyze the inhomogeneous wave equation in the frequency-domain. Hence, by taking the Fourier transform of (3.22) we obtain
\[
\mathbf{\nabla}^2 \tilde{p} + \left( \frac{\omega}{c_0} \right)^2 \tilde{p} = -\left( \tilde{U}_\kappa + \tilde{U}_\rho \right) \tilde{p}, \tag{3.24}
\]
where \( \tilde{p} = \mathcal{F} \{ p \} \) and \( \tilde{U}_\kappa = \left( \frac{\omega}{c_0} \right)^2 \gamma_\kappa(r) \). The total pressure field is a sum of the incident and the scattered pressure fields, i.e. \( \tilde{p} = \tilde{p}_i + \tilde{p}_s \). Because we expect that all scattered waves are outgoing waves, the scattered pressure field satisfies the Sommerfeld radiation condition \( \lim_{r \to \infty} \left( \frac{2}{\omega} - j \frac{\omega}{c_0} \right) \tilde{p}_s = 0 \). The scattering problem can be formulated as the following boundary-value problem:
\[
\begin{cases}
\mathbf{\nabla}^2 \tilde{p} + \left( \frac{\omega}{c_0} \right)^2 \tilde{p} = -\left( \tilde{U}_\kappa + \tilde{U}_\rho \right) \tilde{p} \\
\tilde{p} = \tilde{p}_i + \tilde{p}_s \\
\lim_{r \to \infty} \left( \frac{2}{\omega} - j \frac{\omega}{c_0} \right) \tilde{p}_s = 0.
\end{cases} \tag{3.25}
\]

It is more useful to formulate the scattering problem in (3.25) as an integral equation, rather than a differential equation plus boundary conditions. It can be shown that the model for scattering...
problems in (3.25) corresponds to the Lippmann-Schwinger equation [73]:

\[
\bar{p}(\mathbf{r}, \omega) = \bar{p}_i(\mathbf{r}, \omega) + \int_{\mathbb{R}^3} G(\mathbf{r}'|\mathbf{r}; \omega) \left( \bar{U}_\kappa + U_\rho \right) \bar{p}(\mathbf{r}', \omega) \, dV',
\]

where

\[
G(\mathbf{r}'|\mathbf{r}; \omega) = \frac{e^{-\frac{\omega}{c_0} |\mathbf{r} - \mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|}
\]

is the Green’s function of an unbounded medium in the frequency-domain.

In general, Eq. (3.26) cannot be solved exactly. This equation can be solved by means of successive approximations. If the scattered wave is weak compared to the incident wave, the first approximation in (3.26) gives a satisfactory representation of the scattered field. The first approximation to solve Eq. (3.26), known as the Born approximation, results

\[
\bar{p}(\mathbf{r}, \omega) = \bar{p}_i(\mathbf{r}, \omega) + \int_{\mathbb{R}^3} G(\mathbf{r}'|\mathbf{r}; \omega) \left( \bar{U}_\kappa + U_\rho \right) \bar{p}_i(\mathbf{r}', \omega) \, dV'.
\]

In the Born approximation, we identify the pressure distribution of the scattered wave as

\[
\bar{p}_s(\mathbf{r}, \omega) = \int_{\mathbb{R}^3} G(\mathbf{r}'|\mathbf{r}; t) \left( \bar{U}_\kappa + U_\rho \right) \bar{p}_i(\mathbf{r}', t) \, dV'.
\]

The pressure distribution of the scattered wave in the time-domain can be obtained by taking the inverse Fourier in (3.29). Accordingly,

\[
p_s(\mathbf{r}, t) = \int_{\mathbb{R}^3} g(\mathbf{r}'|\mathbf{r}; t) \left( \bar{U}_\kappa + U_\rho \right) \bar{p}_i(\mathbf{r}', t) \, dV',
\]

where

\[
g(\mathbf{r}'|\mathbf{r}; t) = \frac{\delta(t - |\mathbf{r} - \mathbf{r}'|/c_0)}{4\pi |\mathbf{r} - \mathbf{r}'|}
\]

is the Green’s function in the time-domain.

So far, the scattering theory presented here accounts only for scattering of compressional waves by variations in density and compressibility. Scattering problems involving elastic or viscoelastic homogeneous objects embedded in ideal fluids is much more complex. In these cases, shear wave propagation and attenuation inside the objects should be taken into account.

We are particularly interested in the acoustic scattering due to a point-target. By solving this problem, one can determine the radiation force exerted on a point-target and use the result to define the vibro-acoustography point-spread function. Mathematically, a point-target is a point (zero-dimensional) occupying an infinitesimal volume $dV$ with some mechanical property described by the Dirac delta function. Physically, no object has such characteristics.
Assume that a point-target placed at \( r_0 \) with compressibility and density given by

\[
\begin{align*}
\kappa_V &= \kappa_0 + (\kappa_0 - \kappa_1) \delta^3(r - r_0), \\
\rho_V &= \rho_0 + \frac{2\rho_0(\rho_1 - \rho_0)}{2\rho_1 + \rho_0} \delta^3(r - r_0),
\end{align*}
\tag{3.32}
\]

where \( \kappa_1 \) and \( \rho_1 \) are the compressibility and density in the infinitesimal volume \( dV \) and \( \delta^3(r - r_0) \) is the three-dimensional Dirac delta function. The choice of \( \rho_V \) is based on the load caused by the fluid on a small sphere in a sound field. For further details see Ref. [66]. Notice that when \( r \neq r_0 \) both compressibility and density become \( \kappa_0 \) and \( \rho_0 \) as required in Eq. (3.23). The scattering potentials for the point-target described by (3.32) are given by

\[
\begin{align*}
\widetilde{U}_k &= A_k \left( \frac{\omega}{c_0} \right)^2 \delta^3(r - r_0), \\
U_p &= -A_p \nabla \cdot \left[ \delta^3(r - r_0) \nabla \right],
\end{align*}
\tag{3.33}
\]

where \( A_k = \left( 1 - \frac{\kappa_1}{\kappa_0} \right) \) and \( A_p = \frac{2(\rho_1 - \rho_0)}{2\rho_1 + \rho_0} \). If no target is present in the medium, i.e. \( \kappa_1 = \kappa_0 \) and \( \rho_1 = \rho_0 \), then \( \widetilde{U}_k = U_p = 0 \). Furthermore, when \( \rho_1 \gg \rho_0 \) then \( A_k \sim A_p \sim 1 \). Substituting (3.33) in (3.29) gives us the scattered pressure distribution as follows

\[
\begin{align*}
\bar{p}_s(r, \omega) &= A_k \left( \frac{\omega}{c_0} \right)^2 \bar{p}_1(r_0, \omega) G(r|r_0; \omega) - A_p G(r|r_0; \omega) \nabla^2 \bar{p}_1(r, \omega)\big|_{r=r_0} \\
&\quad - A_p \int_{\mathbb{R}^3} G(r|r'; \omega) \nabla' \bar{p}_1(r', \omega) \cdot \nabla' \delta^3(r' - r_0) \, dV(r'),
\end{align*}
\tag{3.34}
\]

Evaluating the integral in (3.34) by parts and using (2.12) results

\[
\bar{p}_s(r, \omega) = A_k \left( \frac{\omega}{c_0} \right)^2 \bar{p}_1(r_0, \omega) G(r|r_0; \omega) + A_p \left[ \nabla' \bar{p}_1(r', \omega) \cdot \nabla G(r|r_0; \omega) \right]_{r'=r_0}
\tag{3.35}
\]

The scattered pressure distribution has a monopole and a dipole radiation pattern (the first and the second terms, respectively).

The dipole term of (3.35) can be simplified eliminating the gradient of the Green’s function in terms of the prime variables. Let \( r = (x_1, x_2, x_3) \), \( r' = (x'_1, x'_2, x'_3) \), and \( R = |r - r'| \). The following relations hold \( \frac{\partial G}{\partial x'_k} = \frac{\partial G}{\partial x_k} \frac{\partial x'_k}{\partial x_k} \) and \( \frac{\partial^2 G}{\partial x'_k \partial x'_l} = \frac{\partial^2 G}{\partial x_k \partial x_l} \frac{\partial x'_k}{\partial x_k} \frac{\partial x'_l}{\partial x_l} \) for \( k = 1, 2, 3 \). It follows immediately that \( \frac{\partial R}{\partial x'_k} = -\frac{\partial R}{\partial x_k} \), therefore \( \frac{\partial^2 G}{\partial x'_k \partial x'_l} = -\frac{\partial^2 G}{\partial x_k \partial x_l} \). Hence, \( \nabla' G(r|r_0; \omega) = -\nabla G(r|r_0; \omega) \). Taking the Fourier transform of Eq. (2.34) yields \( \nabla \bar{p}_1(r, \omega)\big|_{r=r_0} = j\rho_0 \omega \bar{V}_1(r_0, \omega) \). Therefore, we can write Eq. (3.35) as

\[
\bar{p}_s(r, \omega) = A_k \left( \frac{\omega}{c_0} \right)^2 \bar{p}_1(r_0, \omega) G(r|r_0; \omega) - j\rho_0 \omega A_p \bar{V}_1(r_0, \omega) \cdot \nabla G(r|r_0; \omega). \tag{3.36}
\]
The Fourier transform of Eq. (2.35) gives $\tilde{p} = j\omega\tilde{\rho}\tilde{\phi}$. Using this result, we obtain the acoustic potential of the scattered wave as follows

$$\tilde{\phi}_s(r, \omega) = \frac{jA_\kappa\omega}{\rho_0 c_0^2} \tilde{p}_i(r_0, \omega) G(r|r_0; \omega) - A_\rho \tilde{v}_i(r_0, \omega) \cdot \nabla G(r|r_0; \omega).$$

Figure (3.3) shows the radiation pattern produced by a monopole and a dipole centered at the origin of the system coordinate.

### 3.3 Ultrasound radiation force on a point-target

A vibro-acoustography system is characterized by its PSF which is related to the spatial distribution of the dynamic ultrasound radiation force on a point-target. This force causes a vibration of the point-target. Thus, the dynamic behavior of the point-target at the modulation frequency $\Delta\omega$ should also be analyzed. When an external force is applied to an object that has a restoration mechanism, the object oscillates around an equilibrium point. For example, a spring-mass system, a string of a guitar, a drum, and a simple pendulum. In all cases, there is a restoration mechanism in the system. We assume that the point-target can vibrate at the frequency $\Delta\omega$ around an equilibrium point in the direction of the radiation force. The point-target can be imagined as a spring-mass system. In this case, the acoustic emission of the point-target follows a dipole radiation pattern. Nevertheless, the radiation pattern of the point-target at the frequency $\Delta\omega$ could be a monopole radiation, without changing the main results about the system PSF. The ultrasound radiation force on a point-target will be analyzed for incident monochromatic and modulated ultrasound waves. The former produces a static radiation force. The latter generates a radiation force with a static and a dynamic components.
3.3.1 Monochromatic ultrasound waves

Most studies carried out on acoustic radiation force consider an incident monochromatic plane wave striking a target. Consider an ultrasound wave with frequency $\omega_0$ hitting a point-target described by Eq. (3.32). This wave is described by the following acoustic potential

$$\phi_i(r, t) = \hat{\phi}_0(r) e^{i \omega_0 t}, \quad (3.38)$$

where $\hat{\phi}_0$ is the complex spatial distribution of the wave. Using Eq. (2.18) one obtains the acoustic potential in frequency domain as

$$e^{j \omega_0 t} \phi_i(r, \omega) = 2\pi \hat{\phi}_0(r) \delta(\omega - \omega_0). \quad (3.39)$$

By substituting this equation in (3.37) and taking the inverse Fourier transform, we obtain the potential function of the scattered wave as follows

$$\phi_s(r, t) = \frac{j \omega_0 A \kappa}{\rho_0 c_0^2} \hat{p}_i(r_0, t) G(r | r_0; \omega_0) - A_p \hat{v}_i(r_0, t) \cdot \nabla G(r | r_0; \omega_0) \cos \theta, \quad (3.40)$$

where $p_i$ and $v_i$ are the incident pressure and particle velocity. To simplify the analysis, we consider that the point-target is at the origin of the coordinate system, $r_0 = 0 = (0, 0, 0)$, and the particle velocity $v_i(0, t)$ is in the $z$-direction. In this case, the amplitude of the potential function of the scattered wave in the region $r \to \infty$ can be written as

$$\hat{\phi}_s(r) \sim \frac{A \hat{p}_i}{\rho_0 c_0^2} - \frac{A_p \hat{v}_i \cos \theta}{c_0} \frac{\omega_0 e^{-j(\omega r/c_0 - \pi/2)}}{4\pi r}, \quad (3.41)$$

where $\hat{p}_i$ and $\hat{v}_i$ are the amplitude of the pressure and velocity fields.

Let us examine the radiation force term due to the scattered wave. Consider that the control surface $S$ in Eq. (3.19) is a sphere with a very large radius centered at the point-target. Substituting $v_s = -\nabla \phi_s$ in Eq. (3.19) we have

$$f_{ss} = -\rho_0 \int_S (\nabla \phi_s \cdot e_r) \nabla \phi_s \, dS \sim -2\pi r^2 \rho_0 e_z \int_0^\pi \left| \frac{\partial \phi_s}{\partial r} \right|^2 \cos \theta \sin \theta d\theta,$$

$$= \frac{A \kappa A_p \omega_0^4}{12\pi c_0^5} \text{Re} \{\hat{p}_i \hat{v}_i^*\},$$

$$= \frac{A \kappa A_p \omega_0^4 I}{6\pi c_0^5}, \quad (3.42)$$

where $I = \frac{1}{2} \text{Re} \{\hat{p}_i \hat{v}_i\}$ is the time-averaged intensity vector at the point-target position. Even though
Eq. (3.42) was obtained for a point-target at \( r_0 = \vec{0} \) with the particle velocity \( \mathbf{v}_i \) in the direction of \( \mathbf{e}_z \), the equation is valid for a point-target localized anywhere in the space with an any velocity field. This is because a coordinate system can always be chosen to satisfy the conditions assumed to derive Eq. (3.42). Thus, we conclude that the scattering component of the radiation force is in the direction of the velocity field at the target position.

The radiation force due to the interference between the incident and the scattered waves, Eq. 3.18, can be written as

\[
f_{is} = \rho_0 \int_S \left[ (\mathbf{v}_i \cdot \mathbf{v}_s) - \frac{p_i p_s}{\rho_0 c_0^2} \right] \mathbf{e}_r - \left[ (\mathbf{v}_s \cdot \mathbf{e}_r) \mathbf{v}_i + (\mathbf{v}_i \cdot \mathbf{e}_r) \mathbf{v}_s \right] dS. \quad (3.43)
\]

Changing the surface integral to a volume integral using Eqs. (2.10), (2.13), and (2.11) we obtain

\[
f_{is} = \rho_0 \int_V \left( \frac{1}{\rho_0 c_0^2} \nabla (p_i p_s) + (\nabla \cdot \mathbf{v}_s) \mathbf{v}_i + (\nabla \cdot \mathbf{v}_i) \mathbf{v}_s \right) dV. \quad (3.44)
\]

Using the linear Euler equation (2.34), we can rewrite this expression as

\[
f_{is} = \rho_0 \int_V \left( \left( \nabla \cdot \mathbf{v}_i - \frac{p_s}{\rho_0 c_0^2} \frac{\partial}{\partial t} \right) \mathbf{v}_i + \left( \nabla \cdot \mathbf{v}_s - \frac{p_i}{\rho_0 c_0^2} \frac{\partial}{\partial t} \right) \mathbf{v}_s \right) dV. \quad (3.45)
\]

Since the incident wave is monochromatic, we have

\[
\left( \nabla \cdot \mathbf{v}_i - \frac{p_i}{\rho_0 c_0^2} \frac{\partial}{\partial t} \right) \mathbf{v}_i = \left( \nabla^2 \phi_i - \frac{1}{c_0^2} \frac{\partial^2 \phi_i}{\partial t^2} \right) \mathbf{v}_i = 0,
\]

Thus, Eq. (3.45) turns to

\[
f_{is} = \rho_0 \int_V \left( \left( \nabla \cdot \mathbf{v}_s - \frac{p_s}{\rho_0 c_0^2} \frac{\partial}{\partial t} \right) \mathbf{v}_i \right) dV. \quad (3.47)
\]

This equation can be rewritten in terms of the scattered acoustic potential as follows

\[
f_{is} = -\rho_0 \int_V \left( \left( \nabla^2 \phi_s - \frac{1}{c_0^2} \frac{\partial^2 \phi_s}{\partial t^2} \right) \mathbf{v}_i \right) dV. \quad (3.48)
\]

To evaluate Eq. (3.48) for a point-target at \( r_0 \), we should go back to Eq. (3.40). According to this
equation we have

\[ \nabla^2 \phi_s - \frac{1}{c_0^2} \frac{\partial^2 \phi_s}{\partial t^2} = - \frac{j \omega_0 A K_p_i(r_0, t)}{\rho_0 c_0^2} \left[ \nabla^2 G(r | r_0; \omega_0) + \left( \frac{\omega_0}{c_0} \right)^2 G(r | r_0; \omega_0) \right] \\
+ \ A_p v_i(r_0, t) \cdot \nabla \left[ \nabla^2 G(r | r_0; \omega_0) + \left( \frac{\omega_0}{c_0} \right)^2 G(r | r_0; \omega_0) \right] \\
= \frac{j \omega_0 A K_p_i(r_0, t)}{\rho_0 c_0^2} \delta^3(r - r_0) - A_p v_i(r_0, t) \cdot \nabla \delta^3(r - r_0). \tag{3.49} \]

By substituting Eqs. (3.49) into (3.48) we note that the first term in the resulting expression can be easily integrated. The second term can be integrated by parts and rearranged using Eq. (2.11). After these evaluations, the acoustic radiation force due to the interference term is given by

\[ f_{is} = -\nabla \left[ A \frac{\langle P^2 \rangle}{2 \rho_0 c_0^2} - A_p \frac{\rho \langle v_i \cdot v_i \rangle}{2} \right]. \tag{3.50} \]

The radiation force is given as the gradient of the potential and kinetic energy densities of the incident wave. For a plane traveling wave, the radiation force due to interference is very small compared to the radiation force due the scattering term given in (3.42). Therefore, the main contribution for the radiation force exerted by a plane traveling wave comes from the scattering term of the force. However, this is not true when the target experiences a plane standing wave.

Consider a sphere of radius \( a \), density \( \rho_1 \), and compressibility \( K_1 \), in a plane standing wave with wavelength \( \lambda \), with \( a \ll \lambda \). The main contribution of the radiation force can be calculating simply by integrating \( f_{is} \) in the volume of the sphere. The result is in agreement with the derivation of Gor’kov [41] which has King’s [37] and Yosioka’s [39] theories as particular cases.

### 3.3.2 Modulated ultrasound waves

Dynamic radiation force can be generated by means of an ultrasound modulated wave. As previously mentioned, the ultrasound modulated beam can be produce by a dual or AM-single beam. Here, we focus on dual beam mode because AM-single beam can be treated as a particular case of the former mode.

Consider a modulated incident wave described by the following acoustic potential

\[ \phi_i(r, t) = \hat{\phi}_i^{(a)}(r)e^{i \omega_0 t} + \hat{\phi}_i^{(b)}(r)e^{i \omega_b t}, \tag{3.51} \]

where \( \hat{\phi}_i^{(a)} \) and \( \hat{\phi}_i^{(b)} \) are the complex amplitude functions of the waves \( a \) and \( b \), respectively. The wave
frequencies are $\omega_a = \omega_0$ and $\omega_b = \omega_0 + \Delta \omega$. The superposition principle can be used to solve the scattering problem of waves $a$ and $b$, separately. The resulting scattered wave is given by

$$\phi_s(\mathbf{r}, t) = \hat{\phi}_s(a)(\mathbf{r}) e^{j \omega_0 t} + \hat{\phi}_s(b)(\mathbf{r}) e^{j \omega_0 t},$$

(3.52)

where $\hat{\phi}_s(a)$ and $\hat{\phi}_s(b)$ are the complex amplitude functions of the scattered waves $a$ and $b$, respectively.

The velocity fields of the incident and scattered waves have a contribution of each ultrasound source $a$ and $b$. Accordingly,

$$v_i(\mathbf{r}, t) = \hat{v}_i(a)(\mathbf{r}) e^{j \omega_0 t} + \hat{v}_i(b)(\mathbf{r}) e^{j \omega_0 t},$$

$$v_s(\mathbf{r}, t) = \hat{v}_s(a)(\mathbf{r}) e^{j \omega_0 t} + \hat{v}_s(b)(\mathbf{r}) e^{j \omega_0 t};$$

(3.53)

where $\hat{v}_i(a)(\mathbf{r})$ stands for the complex amplitude of the velocity fields $a$ and $b$.

The radiation force exerted by a modulated ultrasound wave obtained through the short-term time average has a dynamic and two static parts. By calculating the incident and scattered pressure distributions and substituting the result with the velocity fields in (3.53) into (3.47), we get

$$f(t) = f(a) + f(b) + f(d)(t),$$

(3.54)

where $f(a)$ and $f(b)$ are the static radiation force caused by each individual ultrasound wave given in Eq. (3.50) and $f(d)$ is the dynamic component of the radiation force. This term is caused by the interference between the ultrasound beam in the overlapping region. The radiation force given in Eqs. (3.18) and (3.19) is still valid for a modulated incident wave described by (3.51). However, the radiation force should be calculated through the short-term time average defined in (2.25).

The scattering term of the radiation force given by (3.19) can be expanded using Eq. (3.53) as

$$f_{ss}^{(d)} = -p_0 \int_S \left( \mathbf{v}_i(a) \cdot \mathbf{e}_r \right) \mathbf{v}_s(b) + \left( \mathbf{v}_i(b) \cdot \mathbf{e}_r \right) \mathbf{v}_s(a) dS.$$  

(3.55)

We consider the same hypotheses for the point-target and the velocity field as assumed in Sec. 3.3.1. Substituting the acoustic potential $\phi_s(a)$ and $\phi_s(b)$ in this expression, performing a similar derivation as Eq. (3.42), and using Eq. (2.26) we have

$$f_{ss}^{(d)} = A_0 \left[ I^{(a)} + I^{(b)} \right],$$  

(3.56)
Chapter 3: Ultrasound radiation force

where $A_0$ is a constant,

$$
\mathbf{I}^{(a)} = \frac{\text{Re}\left\{ \mathbf{p}^{(b)}_i \mathbf{v}^{(a)*}_i e^{j\Delta\omega} \right\}}{2},
$$

(3.57)

$$
\mathbf{I}^{(b)} = \frac{\text{Re}\left\{ \mathbf{p}^{(a)}_i \mathbf{v}^{(b)*}_i e^{j\Delta\omega} \right\}}{2}
$$

(3.58)

are the time-varying intensity vectors in the direction of $\mathbf{v}^{(a)}_i$ and $\mathbf{v}^{(b)}_i$, respectively. We can rewrite Eq. (3.56) in terms of the acoustic potential amplitudes as follows

$$
\mathbf{f}^{(d)}_{ss} \propto \text{Re}\left\{ \nabla \left[ \hat{\phi}^{(a)}_i \hat{\phi}^{(b)}_i \right] e^{j\Delta\omega} \right\},
$$

(3.59)

Now we focus on the interference term of the radiation force. It can be shown from Eq. (3.45) that the interference term of the dynamic radiation force is given by

$$
\mathbf{f}^{(d)}_{is} = -\rho_0 \left[ \int \left( \nabla \phi^{(b)}_i - \frac{1}{c_0^2} \frac{\partial^2 \phi^{(b)}_i}{\partial t^2} \right) \mathbf{v}^{(a)}_i \right] \mathbf{v}^{(b)}_i dV
$$

(3.60)

Using Eq. (3.49) in this expression, we get

$$
\mathbf{f}^{(d)}_{is} = -\frac{A_k}{\rho_0 c_0^2} \left( \frac{\omega_a}{\omega_b} \frac{\partial}{\partial t} \mathbf{p}^{(a)}_i \cdot \nabla \mathbf{p}^{(b)}_i + \frac{\omega_b}{\omega_a} \frac{\partial}{\partial t} \mathbf{p}^{(a)}_i \cdot \nabla \mathbf{p}^{(b)}_i \right)_T + \rho_0 A_p \left\langle \mathbf{v}^{(a)}_i \cdot \mathbf{v}^{(b)}_i \right\rangle_T.
$$

(3.61)

This component of the radiation force can be further simplified by expanding the terms $\frac{\omega_a}{\omega_b}$ and $\frac{\omega_b}{\omega_a}$, with $\frac{\Delta\omega}{\omega_b} \ll 1$. Accordingly, we find

$$
\mathbf{f}^{(d)}_{is} = -\nabla \left\langle \frac{A_k p^{(a)}_i p^{(b)}_i}{2 \rho_0 c_0^2} - \frac{A_p \rho_0 \left( \mathbf{v}^{(a)}_i \cdot \mathbf{v}^{(b)}_i \right)}{2} \right\rangle_T.
$$

(3.62)

This term becomes very small for plane or quasi-plane progressive waves if the angle between $\mathbf{v}^{(a)}_i$ and $\mathbf{v}^{(b)}_i$ is small. If this angle is large or the incident ultrasound field is a standing wave, the contribution of Eq. (3.62) to the dynamic radiation force should be taken into account.

Consider a collimated ultrasound beam formed by two quasi-plane progressive wave propa-
gating in the $z$-direction described by Eq. (3.51). The amplitude of the acoustic potentials are

$$\hat{\phi}_i^{(a)}(r) = \hat{\phi}_i^{(a)}(r)e^{-\omega_0 z/c_0}, \quad (3.63)$$

$$\hat{\phi}_i^{(b)}(r) = \hat{\phi}_i^{(b)}(r)e^{-\omega_0 z/c_0}, \quad (3.64)$$

where $\hat{\phi}_i^{(a)}$ and $\hat{\phi}_i^{(b)}$ are the complex amplitudes of the ultrasound waves. We assume that the transverse variation of the potentials given in (3.63) and (3.64) is much smaller than that in the axial direction and also the conditions

$$\frac{\partial \hat{\phi}_i^{(a)}}{\partial z} \simeq -\frac{j\omega_a}{c_0} \hat{\phi}_i^{(a)}; \quad (3.65)$$

$$\frac{\partial \hat{\phi}_i^{(b)}}{\partial z} \simeq -\frac{j\omega_b}{c_0} \hat{\phi}_i^{(b)}; \quad (3.66)$$

If these hypotheses hold, the main contribution for the dynamic radiation force comes from Eq. (3.56). Therefore

$$f^{(d)}(t) \propto \text{Re} \left\{ \hat{\phi}_i^{(a)} \hat{\phi}_i^{(b)*} e^{j\Delta \omega t} \right\} e_z. \quad (3.67)$$

The dynamic radiation force exerted by a collimated quasi-plane wave on a point-target is proportional to the product of the spatial amplitude distribution of the ultrasound waves. This result is similar to Eq. (2.51) which describes the radiation force on a planar target.

### 3.4 Summary

The theory of ultrasound radiation force for vibro-acoustography was developed based on the radiation-stress tensor. The radiation force exerted on a point-target by any incident ultrasound wave was calculated by solving the acoustic scattering problem. Results show that in dual beam mode the radiation force on a point-target is proportional to the sum of the intensity vectors of each ultrasound beam. The presented radiation force theory generalizes the quasi-plane wave model for vibro-acoustography, in which only the component of the force parallel to the wave propagation direction is considered.
Chapter 4

Beamforming and image formation in vibro-acoustography

Imaging techniques based on ultrasound waves such as ultrasonography and vibro-acoustography illuminate the region of interest by means of focused ultrasound beams. To form a picture, the ultrasound beam should scan the region of interest. The scanning process can be accomplished either by mechanically moving the transducer or electronically steering the beam by using array transducers. Beamforming in vibro-acoustography involves the formation of continuous-wave (CW) ultrasound focused beams. The theory of ultrasound radiation from baffled pistons based on linear acoustics is used to model the generation of the required ultrasound beams in vibro-acoustography. Image synthesis in vibro-acoustography depends upon the acoustic emission of the region of interest. We present a simple model for the acoustic emission of hard small object which follows a dipole radiation pattern. The point-spread function (PSF) of vibro-acoustography systems based on spherical confocal and sector transducers is derived based on the theory of the ultrasound radiation force presented in Chapter 3.

4.1 Ultrasound beamforming

An ultrasound transducer can be modeled as a piston mounted in an infinite extended planar rigid baffle. It is assumed that the baffle is immovable, hence only the piston can move in its normal direction (see Figure 4.1). The pressure field radiated by a piston mainly depends on its geometric shape and the normal velocity distribution applied on the piston surface. Consider a problem of determining the pressure field at a spatial point in the half-space $z \geq 0$ resulting from the radiation of a piston $\Omega$ mounted in a baffled plane surface at $z = 0$. The problem is formulated as a classical boundary-value problem to solve the linear wave equation (2.36) for the acoustic potential $\phi(r,t)$. The mathematical
Figure 4.1: An arbitrary shaped piston with surface $\Omega$ in a rigid baffle.

specification of the boundary-value problem yields the following system of equation

$$
\begin{cases}
\nabla^2 \phi(r, t) - \frac{1}{c_0^2} \frac{\partial^2 \phi(r, t)}{\partial t^2} &= 0, \quad r \in \{ \mathbb{R}^3 \mid z > 0 \}, \\
\frac{\partial \phi(r, t)}{\partial z} &= -v_n(r, t), \quad r \in \Omega, \ t > 0, \\
\phi(r, 0) &= \frac{\partial \phi(r, t)}{\partial t} \bigg|_{t=0} = 0, \quad r \in \{ \mathbb{R}^3 \mid z > 0 \},
\end{cases}
$$

(4.1)

where $v_n$ is the normal component of the velocity field on the piston. Based on the linear system theory and the time-dependent Green’s function, Stepanishen [62] obtained a solution of (4.1) as the time convolution integral

$$
\phi(r, t) = v_n(t) * h(r, t),
$$

(4.2)

where $h(r, t)$ is the spatial impulse response function of the piston. The impulse response function is defined as

$$
h(r, t) = \int_{\Omega} \frac{\delta(t - |r - r'|/c_0)}{2\pi |r - r'|} dS(r').
$$

(4.3)

The spatial impulse function $h(r, t)$ is time limited, amplitude bounded, and piecewise continuous in time. The time duration of $h(r, t)$ is typically the time that the acoustic wave takes to transit across the piston surface. Several methods have been proposed to calculate $h(r, t)$ numerically for arbitrary shaped pistons [79, 80, 81].

Vibro-acoustography applications require transducers driven by CW-signals. Consider that the normal component of the velocity on the surface of the piston is given by

$$v_n(t) = v_0 e^{j \omega t},
$$

(4.4)
where $v_0$ is the amplitude of the oscillation and $\omega_0$ is the angular frequency. Substituting (4.4) into (4.3), using $v_0 = \left(\frac{2I}{\rho_0 c_0}\right)^{1/2}$ and the Fourier transform on (4.3), one can show that

$$\phi(r,t) = \left(\frac{2I}{\rho_0 c_0}\right)^{1/2} H(r, \omega_0) e^{j\omega_0 t},$$

where $H(r, \omega_0)$ is given by the Fourier transform of the spatial impulse function at the frequency $\omega_0$ as follows

$$H(r, \omega_0) = \int_{-\infty}^{+\infty} h(r,t)e^{-j\omega_0 t} dt. \quad (4.6)$$

By evaluating this integral we can rewrite Eq. (4.5) as

$$\phi(r,t) = \left(\frac{2I}{\rho_0 c_0}\right)^{1/2} e^{j\omega_0 t} \int_{\Omega} \frac{e^{-j\omega_0 |r-r'|/c_0}}{2\pi|\mathbf{r}-\mathbf{r}'|} dS(r'). \quad (4.7)$$

This is the Rayleigh integral [82]. Eq. (4.7) can be understood as the Huygens’ principle in which every element $dS$ on the surface $\Omega$ is considered a point-source emitting a spherical wave.

### 4.2 Acoustic emission

An object that is submitted to a dynamic force responds to the force according to its mechanical properties and boundary conditions. A simple example is tapping a drum with a time-varying force. The mechanical properties of the drum’s membrane such as elasticity and density produce a specific type of sound. The boundary condition of the drum is related to its geometric shape. A drum with different shape may produce a different sound pattern as well as a different tension in the mem-
brane can produce different sounds in pitch. Furthermore, the position where the force is applied also influences the sound produced by the membrane.

The example of the drum can be adapted to understand the response of objects to the dynamic radiation force. In ultrasound vibro-acoustography, a dynamic force distributed on less than 1 mm$^2$ transversely “taps” an object embedded in the propagating medium. In response, the object vibrates according to its mechanical properties and boundary conditions. The object emits an acoustic field in the propagating medium which can be detect some distance away. The detected signal has information of the object mechanical properties and boundary conditions. This signal is used to synthesize an image of the object.

We are interested in calculating the acoustic emission of a point-target. Consider that the point-target behaves as an infinitesimal dipole source. The vibrating point-target needs to be connected to some sort of restoration force mechanism. Assume a small sphere suspended by a string forming a simple pendulum. The acoustic emission by a simple pendulum could be approximated to a dipole radiation pattern if the oscillation amplitude of the pendulum is small. In this case, the restoration force is the gravitational force.

Consider a infinitesimal dipole in an infinite medium placed at the origin of the coordinate system as described by Fig. 4.3. The radiation force responsible for the dipole vibration is in the direction of the $z$-axis. The dipole produces a pressure field given by [65]

$$p = \hat{f} \cos \theta \frac{\partial}{\partial r} \left[ \frac{e^{j\omega(t-r/c_0)}}{4\pi r} \right],$$

(4.8)

where $\hat{f}$ is the amplitude of the magnitude of the dynamic component of the radiation force, which is the total force needed to move the point-target in transatory fashion. The far-field of the dipole radiation pattern, $\frac{\Delta \omega}{c_0} r \gg 1$, is dominated by monopole radiation which decrease as $1/r$. Consider that
the acoustic emission is detected in the far-field region. In this case, Eq. (4.8) becomes

\[ p = \frac{j\Delta\omega \hat{f} \cos \theta e^{j\Delta\omega t/r_0}}{4\pi r} \]. \hspace{1cm} (4.9)

Now, we can generalize Eq. (4.9) for a point-target in an arbitrary position \( r_1 \). In this case, the amplitude of the acoustic emission measured at \( r_2 \) is given by

\[ p(r_2|r_1) = \frac{j\Delta\omega}{c_0} \hat{f}(r_1) \cos \theta_1 G(r_2|r_1;\Delta\omega), \hspace{1cm} (4.10) \]

where \( \theta_1 \) is the angle between the direction of the radiation force with \( r_2 \) and \( G(r_2|r_1;\Delta\omega) \) is the Green’s function given in Eq. (3.27). This expression has the same form as Eq. (2.52). Notice that the function \( Q \) presented in Eq. 2.52 here becomes constant: \( Q = \frac{\Delta\omega}{c_0} \).

The intensity of the acoustic emission is given by

\[ I = \frac{1}{3} \left( \frac{\Delta\omega}{c_0} \right)^2 |\hat{f}|^2. \hspace{1cm} (4.11) \]

This result shows that higher difference frequencies \( \Delta\omega \) produce stronger emitted signals.

### 4.3 Vibro-acoustography image synthesis

An imaging system is commonly described through its point-spread function (PSF), which describes the degree that a point-target is blurred on the imaging plane. To form an two-dimensional image of an object we need to define an imaging plane and an object function. The image of an object is obtained by convolving the system PSF with the object function on the imaging plane. In vibro-acoustography, this convolution is given by Eq. (2.56). The PSF determines the spatial resolution limit of the system. Physical characteristics of the imaging technique defines the PSF of the system.

Conventional ultrasound imaging systems are usually based on transmitted pulses in tissue. The resulting scattered waves by tissue inhomogeneities are detected by sensors and processed to form the image. The pulse duration determines the depth resolution of the system, which is the resolution along the beam propagation direction (A-line). The transversal dimensions of the ultrasound pulse determines the transversal resolution of the system.

In vibro-acoustography, both depth and transversal resolution are related to the spatial distribution of the system PSF in the three-dimensional space. Consider a vibro-acoustography system with two ultrasound beams focused at \( r_0 \) with amplitude of the acoustic potential described by \( \hat{\phi}_a(r|r_0) \) and \( \hat{\phi}_b(r|r_0) \). The ultrasound radiation force depends also on the focal point \( r_0 \). For simplicity, we
consider the amplitude of the ultrasound radiation force as a complex quantity. Hence, the complex amplitude of the dynamic component of radiation force on a point-target at \( \mathbf{r} \) can be written as

\[
\hat{f}(\mathbf{r}|\mathbf{r}_0) = \hat{f}_\perp(\mathbf{r}|\mathbf{r}_0)e_\perp + \hat{f}_\parallel(\mathbf{r}|\mathbf{r}_0)e_\parallel;
\]

(4.12)

where \( e_\perp \) and \( e_\parallel \) are the unit vectors in the transverse and axial directions of the resulting ultrasound beam, respectively. The functions \( \hat{f}_\perp \) and \( \hat{f}_\parallel \) are the complex amplitudes of the radiation force in the transverse and axial directions (depth), respectively. These functions can be obtained from Eqs. (3.56) and (3.62). In the three-dimensional model for the ultrasound radiation force, the vibro-acoustography PSF can be defined following the same steps as described in Sec. (2.3.2). However, that model is based only in the axial component of the ultrasound radiation force. Here, we defined the vibro-acoustography PSF based on the three-dimensional radiation force as follows,

\[
h_{\text{psf}} = \frac{\hat{f}(\mathbf{r}|\mathbf{r}_0)}{\hat{f}(\mathbf{r}_0|\mathbf{r}_0)},
\]

(4.13)

where

\[
\hat{f} = \sqrt{\hat{f}_\perp^2 + \hat{f}_\parallel^2}.
\]

(4.14)

### 4.4 Spherical concave transducer

High localized pressure fields can be produced by spherical concave transducers. Figure 4.2 shows a spherical concave transducer seen in the axial plane. The acoustic potential produced by the radiation of a spherical concave transducer of radius \( a \) and focal distance \( z_0 \) can be easier treated if \( a \ll z_0 \), which corresponds to Fresnel approximation. In this approximation, Eq. (4.7) can be used to calculate the acoustic fields generated by the concave transducer. Nevertheless, Eq. (4.7) is strictly true only for the potential of a flat transducer in a planar rigid baffle. If the transducer is curved, the integral neglects the fact that waves radiated from any part of the surface are diffracted by other parts of the transducer. This secondary diffraction effect is relatively unimportant if the surface \( \Omega \) is slightly curved and its dimensions are much larger than the wavelength (see Ref. [83] for further details). To avoid the difficulties presented by the secondary diffraction, we assume that the ultrasound beam initially follows a geometric ray path normal to the surface of the transducer. Thus, we can consider new ultrasound sources located at the baffle plane. Using this approximation, the amplitude of acoustic
potential produced by the spherical concave transducer in the far-field region is given by [84]

\[ \hat{\phi}(r,z) = \left( \frac{2I}{\rho_0 c_0} \right)^{1/2} \exp \left[ -j\omega_0 (z + r^2/2z)/c_0 \right] \frac{1}{2\pi z} \int_0^a \int_0^{2\pi} \exp \left[ -j\frac{\omega_0 r' r}{2c_0 z_0} \left( \frac{z_0}{z} - 1 \right) \right] \exp \left( \frac{j\omega_0 r' \cos \theta'}{c_0 z} \right) r' dr' d\theta', \]  

(4.15)

where \( r = \sqrt{x^2 + y^2} \). The potential on the axis of the transducer is given by setting \( r = 0 \) in (4.15). Thus,

\[ \hat{\phi}(0,z) = \frac{\pi a^2}{2z} \left( \frac{2I}{\rho_0 c_0} \right)^{1/2} \sin c \left[ \frac{a}{2\lambda_0} \left( \frac{z_0}{z} - 1 \right) \right] \exp \left[ -j\frac{\omega_0}{c_0} \left( z + \frac{a^2}{4} \left( \frac{1}{z} - \frac{1}{z_0} \right) \right) \right], \]

(4.16)

where \( \sin c(x) = \frac{\sin(\pi x)}{\pi x} \) and \( \lambda_0 = \frac{2\pi \omega_0 \rho_0}{\rho_0 c_0} \), which can be interpreted as a scaled wavelength by the ratio \( a/z_0 \).

In the focal plane \( z = z_0 \), the potential is given according to Eq. (4.15) by

\[ \hat{\phi}(r,z_0) = \frac{\pi a^2}{z_0} \left( \frac{2I}{\rho_0 c_0} \right)^{1/2} \text{jinc} \left( r/\lambda_0 \right) \exp \left[ -j\frac{\omega_0}{c_0} (z_0 + r^2/2z_0) \right], \]

(4.17)

where \( \text{jinc}(x) \equiv J_1(2\pi x)/\pi x \), with \( J_1(x) \) being the first-order Bessel function of first-kind. The ampli-
Figure 4.5: Axial component of the dynamic radiation force on the focal plane of a confocal transducer. The arrows represent the transverse radiation force vector.

The magnitude of the radial component of the velocity particle on the focal plane is given by

$$\hat{v}_r(r, z_0) \simeq \frac{j\omega_0 r}{c_0 z_0} \phi(r, z_0).$$  (4.18)

Differentiating Eq. (4.15) with respect to $z$ and performing the integration, we obtain the axial component of the velocity particle for $z = z_0$ as follows

$$\hat{v}_z(r, z_0) \simeq \frac{j\omega_0}{c_0} \phi(r, z_0).$$  (4.19)

In the neighborhood of the focal point the axial component of the velocity field is larger than the radial component.

4.4.1 Confocal array transducer

For vibro-acoustography applications, confocal array transducers should have at least two spherical concentric elements, which permits the system to operate in either dual or single beam modes. Figure 4.4 shows a two-element confocal transducer whose inner and outer radii are specified by $a_1$ and $a_2$, respectively. The radius of curvature (focal distance) of the transducer is $z_0$. In single beam mode, both elements are driven by the same modulated signal. Because the ultrasound beams produced by spherical concave transducers is better focused transversely than in the depth, we shall
analyzed the vibro-acoustography PSF on the focal plane. In fact, the present vibro-acoustography system produces C-scan images in which the imaging plane is the transducer focal plane.

To analyze the PSF of a dual beam confocal transducer, we need to calculate the acoustic potential of each element in the focal plane. By using the superposition principle, the amplitude of the acoustic potential produced by each element of the transducer is given by

\[
\hat{\phi}^{(a)}(r) = \frac{\pi a_1^2}{z_0} \left( \frac{2I}{\rho_0 c_0} \right)^{1/2} \text{jinc} \left( r/\kappa_1^{(a)} \right) e^{j\omega_a \psi_a}, \\
\hat{\phi}^{(b)}(r) = \frac{\pi}{z_0} \left( \frac{2I}{\rho_0 c_0} \right)^{1/2} \left[ a_2^2 \text{jinc} \left( r/\kappa_2^{(b)} \right) - a_1^2 \text{jinc} \left( r/\kappa_1^{(b)} \right) \right] e^{j\omega_b \psi_b},
\]

where \( \kappa_1^{(a)} = \frac{2\pi c_0}{\omega a_1^2} \), \( \kappa_1^{(b)} = \frac{2\pi c_0}{\omega b a_1^2} \), \( \kappa_2^{(b)} = \frac{2\pi c_0}{\omega b a_2^2} \), \( \psi_a = \exp \left[ -j\omega_a (z_0 + r^2/2z_0)/c_0 \right] \), and \( \psi_b = \exp \left[ -j\omega_b (z_0 + r^2/2z_0)/c_0 \right] \).

The main contribution for the dynamic radiation force produced by the confocal transducer comes from the scattering term of the force. According to Eqs. (3.56), (3.57), and (3.58) the amplitude
of the axial and transverse components of the dynamic radiation force on the focal plane are given by

\[
\hat{f}_\parallel \propto \hat{\phi}^{(a)}(r, z_0)\hat{\phi}^{(b)*}(r, z_0),
\]

\[
\hat{f}_\perp \propto \nabla_\perp \left[\hat{\phi}^{(a)}(r, z)\hat{\phi}^{(b)*}(r, z)\right]_{z=z_0} = \frac{c_0}{2\omega_0} \nabla_\parallel \hat{f}_\parallel|_{z=z_0},
\]  

where \(\nabla_\perp\) is the transverse gradient with respect to the \(z\)-direction.

We evaluated the dynamic radiation force fields for a confocal transducer with \(a_1 = 15\) mm, \(a_2 = 22\) mm, and \(z_0 = 70\) mm in water \((c_0 = 1500\) m/s). The numerical calculations were performed with MATLAB 6.2 (MathWorks, Inc.). The elements are driven with frequencies of about 3 MHz. Figure 4.5 shows the axial component of the dynamic radiation force on the focal plane with the transverse force vector overlapped. Figure 4.6 shows the transverse radiation force vector field on the focal plane. The force tends to contract and expand radially in the neighborhood of the focal point. The amplitude of the transverse component of the force is about 10.9\% \((-19.2\) dB) of that in the axial direction. Notice that in the neighborhood of the first zero of the axial force, the transversal force is dominant. This means that a small particle placed at the first zero of the axial force would be subject to the transversal force only. Furthermore, close to the peak of the first sidelobe the transversal radiation force flips its phase.

In the Figure 4.7, we have plots of the PSF based on the total and the axial radiation force. The results are practically the same. However, in the total force model the PSF does not go to zero.
passing from the mainlobe to the first sidelobe, as the PSF based on the axial radiation force does. The transverse resolution of the system in the focal plane is about $10 \times 10$ mm. Sidelobe levels are under $-19$ dB. The first sidelobe is dephased by $180^\circ$ with respect to the mainlobe. Note that the sidelobes of the same transducer operating in AM-single beam is about $-35$ dB.

**4.4.2 Sector array transducer**

The confocal transducer operating in dual beam mode produces high localized ultrasound radiation force. Another way to produce a high localized ultrasound radiation force is to use sector array transducers. This transducer is composed of $N$ spherical concave sectors (radial slices) which can be driven independently. Figure 4.8 shows a sector array transducer with aperture radius $a$ and focal distance $z_0$.

Sector array transducers are a particular case of sector-vortex phased array transducers. A sector-vortex transducer consists of concentric-ring phased array subdivided into sectors. With appropriate phasing, this transducer can produce power absorption patterns useful for hyperthermia cancer therapy [85]. For this purpose, the transducer produces variable diameter annular rings in the focal region. Such focal rings can be effective in “cooking” some tumors if directed around the tumor periphery. Konofagou et al. [18] used a sector array transducer to study tissue stiffness variation with temperature through vibro-acoustography.

Consider a sector array transducer with $N$ elements. The amplitude of the acoustic potential radiated by a symmetric pair of elements in a sector transducer (see Figure 4.8.c) in the focal plane ($z = z_0$) can be described by Eq. (4.15). However, notice that the integral over $\theta'$ is split in two integrals over the intervals $[\theta_n - \theta, \theta_n - \theta + \frac{2\pi}{N}]$ and $[\theta_n + \pi - \theta, \theta_n + \pi - \theta + \frac{2\pi}{N}]$, where $\theta_n$ is the offset angle...
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0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

\[ \hat{\phi}_n(r, \theta) = \left( \frac{2I}{\rho_0 c_0} \right)^{1/2} \exp \left[ -j \omega_0 (z_0 + r^2/2z_0)/c_0 \right] \]
\[ \times \int_0^a dr' r' J_0 \left( \frac{\omega_0 r'}{c_0 z_0} \cos(\theta') \right) \frac{2}{\pi} d\theta' \left[ e^{j \omega_0 r' \cos(\theta' + \theta_n - \theta)/c_0} + e^{-j \omega_0 r' \cos(\theta' + \theta_n - \theta)/c_0} \right]. \]

Using the Jacobi-Anger expansion [67]

\[ e^{jz\cos \theta} = J_0(z) + 2 \sum_{m=1}^{\infty} J_m(z) \cos(m \theta), \]

where \( J_m \) is the \( m \)-th order Bessel function of first-kind, and the relation \( J_m(-z) = (-1)^m J_m(z) \) we get

\[ \hat{\phi}_n(r, \theta) = 2 \left( \frac{2I}{\rho_0 c_0} \right)^{1/2} \exp \left[ -j \omega_0 (z_0 + r^2/2z_0)/c_0 \right] \]
\[ \times \int_0^a dr' r' \left[ \frac{2\pi}{N} J_0 \left( \frac{\omega_0 r'}{c_0 z_0} \right) + 4 \sum_{m=1}^{\infty} \frac{(-1)^m}{2m} s_{2m}(\theta) J_{2m} \left( \frac{\omega_0 r'}{c_0 z_0} \right) \right], \]

where

\[ s_{mn}(\theta) = \cos \left[ m \left( \frac{\pi}{N} + \theta_n - \theta \right) \right] \sin \left( \frac{m\pi}{N} \right). \]
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Using the expression [86]

\[
\int_0^z t J_m(t) dt = z J_{m+1}(z) + 2m \sum_{l=0}^{\infty} J_{m+2(l+1)}(z),
\]

we finally obtain

\[
\hat{\phi}_n(r, \theta) = \frac{2a^2}{\pi z_0} \left( \frac{2I}{\rho_0 c_0} \right)^{1/2} \exp \left[ -j \omega_0 (z_0 + r^2/2z_0)/c_0 \right]
\]

\[
\times \left[ \frac{\pi^2}{N} \text{sinc} \left( \frac{r}{\lambda_0} \right) + \lambda_0 \sum_{m=1}^{\infty} \frac{(-1)^m}{m} s_{(2m)n}(\theta) \left( \frac{J_{2m+1}(r/\lambda_0)}{r} \right) \right]
\]

\[
+ \frac{4m \lambda_0}{N} \sum_{l=0}^{\infty} J_{2(m+l+1)} \left( \frac{r/\lambda_0}{r^2} \right) \right].
\]

Some aspects of the potential function of two symmetric elements in a sector array can be pointed out: the function is symmetric around the polar angle \( \theta \) and the leading term of the function is proportional to the potential function of a spherical concave transducer.

In the axial direction, the acoustic potential of the two symmetric elements is given by

\[
\hat{\phi}_n(0, z) = \frac{\pi a^2}{Nz} \left( \frac{2I}{\rho_0 c_0} \right)^{1/2} \text{sinc} \left[ \frac{a}{2 \lambda_0} \left( \frac{z_0}{z} - 1 \right) \right] \exp \left[ -j \omega_0 \left( \frac{z + a^2}{4} \left( \frac{1}{z} - \frac{1}{z_0} \right) \right) \right],
\]

This expression is identical to Eq. (4.16) except for the factor \( 1/N \).

Now consider a sector array transducer operating in dual beam mode. The amplitude of the acoustic potential produced by the transducer on the focal plane is given by

\[
\hat{\phi}(r, \theta, z_0) = \sum_{n=1}^{N/2} \left[ \hat{\phi}^{(a)}_{2n-1}(r, \theta, z_0) + \hat{\phi}^{(b)}_{2n}(r, \theta, z_0) \right],
\]

where the potentials \( \hat{\phi}^{(a)}_{n} \) and \( \hat{\phi}^{(b)}_{n} \) are given by Eq. (4.24) changing \( \omega_0 \) for \( \omega_a \) and \( \omega_b \), respectively. Similarly to the confocal transducer, the axial and transverse components of the dynamic radiation force on the focal plane are given by Eqs. (4.22) and (4.23). Using Eq. (4.26) the components can be
Figure 4.10: Transverse component of the dynamic radiation force on the focal plane of a sector array transducer. The arrows represent the transverse radiation force vector. The magnitude is normalized by the peak value of the axial component of the force.

written as

\[
\hat{f}_\parallel \propto \sum_{m,n=1}^{N/2} \Phi_{2m-1}(r,z_0)\Phi_{2n}^*(r,z_0),
\]

\[
\hat{f}_\perp \propto \nabla_\perp \left[ \sum_{m,n=1}^{N/2} \Phi_{2m-1}(r,z)\Phi_{2n}^*(r,z) \right]_{z=z_0}.
\]

\[
= \frac{c_0}{2\omega_0} \nabla \parallel f_k \big|_{z=z_0}.
\]

We evaluated the axial and the transversal components of the radiation force produced by a sector transducer with \( N = 8 \), radius \( a = 23.5 \) mm, and focal distance \( z_0 = 75 \) mm in water \((c_0 = 1500 \text{ m/s})\). Each group of elements has frequencies \( f_a = 2 \text{ MHz} +5 \text{ kHz} \) and \( f_b = 2 \text{ MHz} -5 \text{ kHz} \). Eqs. (4.27) and (4.29) were implemented in MATLAB 6.2. Figure 4.9 shows the axial radiation force on the focal plane with the transverse radiation force vector overlapped. In the mainlobe, the transverse radiation force points radially to the focal point. The axial radiation force has 8 sidelobes circularly distributed around the mainlobe. Sidelobes are separated by \( 22.5^\circ \) which corresponds to the spatial distribution of the elements in the transducer. In each sidelobe, the transverse radiation force points to the sidelobe peak. The magnitude of the transverse radiation force is shown in Figure 4.10. The magnitude of this force is about 8\% \((-21.9 \text{ dB})\) of the magnitude of the axial radiation force.
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The system PSF in the lateral direction can be seen in Figure 4.11. Sidelobes are under $-15.5$ dB. The transverse resolution of the system is about $(1.6 \times 1.6)$ mm. Simulations show that for $N = 16$ sidelobes are under $-34$ dB.

### 4.5 Experimental measurements of the system point-spread function

An experimental setup to measure both axial and transverse radiation force components is shown in Figure 4.12. A clamped steel wire of 300 $\mu$m diameter is used as a target for the radiation force. Two laser vibrometers arranged perpendicularly measure the axial and transverse vibration of the wire. A confocal transducer with $a_1 = 15$ mm, $a_2 = 22$ mm, and $z_0 = 70$ mm. The transducer operates in dual beam mode at 3 MHz with difference frequency at 4.61 kHz. The transducer scans the axial plane.

Figure 4.14 shows the measured axial and transverse radiation force in depth (the axial plane). Figure 4.15 shows plots the radiation force components on a line passing in the transducer focal point in lateral direction. The theoretical result for the dynamic radiation force on the wire is
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Figure 4.13: Experimental setup for measure the axial radiation force on a small sphere. A laser vibrometer detects the vibration of the sphere.

obtained by integrating Eqs. (4.22) and (4.23) along the wire. The ratio between the measured transverse and axial components of the radiation force is about 11%. The result is in good agreement with the theoretical prediction. However, the matching between the theoretical and experimental data is more qualitative than quantitative. We can observe that the measured components of the radiation force are broader than their theoretical counterpart. The transverse component exhibits an asymmetry compared to the theoretical result. Some factors may cause this asymmetry: misalignment between the lasers, the wire, or the ultrasound beam; asymmetry in the used confocal transducer; and the non-symmetric boundary conditions of the clamped wire. Chen et al. [70] reported asymmetric measurements of the axial component of the dynamic radiation force on a small sphere. Accordingly, the source of the error laid in the asymmetry of the ultrasound beam generated by the used transducer.

Another experimental setup was used to measure the axial component of the dynamic radiation force due to a sector transducer. A small steel sphere of about 300 µm radius attached to a latex sheet is used as a point-target. A laser vibrometer measures the axial amplitude of the sphere vibration. The experimental setup is described in Figure 4.13. Figure 4.16 shows the measured axial component of the dynamic radiation force produced by an 8-element sector array transducer of 23.5 mm radius. The transducer operates in dual beam mode at 2 MHz with difference frequency at 10 kHz. The experimental result are in good agreement with the theory shown in Figure 4.9. Sidelobes are under −16 dB distributed radially spots separated by 22.5° around the mainlobe.
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4.6 Summary

In this chapter, we applied the radiation force theory developed in Chapter 3 to analyze the beamforming of vibro-acoustography system. The ultrasound beam produced by spherical concave and sector transducers was calculated on the focal plane stemming from the Rayleigh integral. Theoretical results were validated with experiments to measure the ultrasound radiation force. The experimental results are in good agreement with the theory. However, the measured axial and transversal components of the force produced by a concave transducer are still not entirely fitted by the theoretical predictions. Specially asymmetry in the transversal radiation force which can be linked to some sort of asymmetry in the ultrasound beam generated by the transducer.
Figure 4.15: Measured radiation force in lateral direction on the transducer focal plane. (a) Axial radiation force. (b) Transverse radiation force.

Figure 4.16: Experimental result of the axial component of the dynamic radiation force on the focal plane of an 8-element sector array transducer.
Chapter 5

Linear arrays for vibro-acoustography

In this chapter, we present linear array transducer design for vibro-acoustography systems. Firstly, we analyze if the resolution cell based on the axial component of the dynamic radiation force gives a good description of the system. Effects of beam steering and focusing on the resolution cell are analyzed based on computer simulations for six different linear array transducer configurations.

5.1 Linear arrays in conventional ultrasound systems

Clinical applications of ultrasound systems require electronic beam focusing and steering. Linear array transducers are widely used for this purpose [87], because they can focus and steer the ultrasound beam laterally by electronically delaying the signals in the array elements.

In conventional ultrasound imaging, the goals in linear array beamforming have been to achieve narrow beams with low sidelobes and suppressed grating lobes. Moreover, linear array design is limited by probe size and the number of array elements. Suppression of grating lobes is achieved by reducing the interelement distance to less than half of the wavelength. The sidelobes can be reduced by weighting the signal amplitudes of the array elements (apodization technique). Linear array beamforming for vibro-acoustography shares similarities with its counterpart in conventional ultrasound (B-scan).

Two array types are commonly used in ultrasound imaging systems [88]: switched and phased linear arrays. Switched linear arrays have a large number of elements inline. The beam is laterally incremented by selecting sequential sub-groups of the array elements. This configuration is suitable, for example, to produce rectangular B-scan images. Phased arrays produce a beam steered in a sector format and sector shaped images in B-scan mode. These two configurations of linear array transducers are shown in Figure 5.1. The imaging planes for B- and C-scan images are the $xz$-plane (range plane) and $xy$-plane (elevation plane), respectively.
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5.2 Linear arrays for vibro-acoustography

Consider a transducer with two linear arrays, a and b, for a vibro-acoustography system. Each linear array has N elements. The elements of the arrays are excited by two CW-signals with proper phase values to focus and steer the resulting ultrasound beam. The acoustic potential resulting from two transducers driven by frequencies $\omega_a$ and $\omega_b$ is given by Eq. (2.46). In the case of two linear
Figure 5.3: Dual inline linear array transducers with $N = 4$. Each transducer has two linear arrays $a$ and $b$. (a) symmetrically separated, (b) centered, and (c) interlaced. All arrays have pitch $d$ and the elements have width $w$ and height $l$.

array transducers, the dimensionless amplitude functions $\hat{\phi}_a$ and $\hat{\phi}_b$, for the beams focused at $r_0$, are given by

$$\hat{\phi}_a(r|r_0) = \frac{\omega_a}{c_0} \sum_{n=1}^{N} a_n e^{-j\psi_a^{(n)}(r_0)} H \left(r - \mathbf{r}_a^{(n)}, \omega_a\right), \quad (5.1)$$

$$\hat{\phi}_b(r|r_0) = \frac{\omega_b}{c_0} \sum_{n=1}^{N} a_n e^{-j\psi_b^{(n)}(r_0)} H \left(r - \mathbf{r}_b^{(n)}, \omega_b\right), \quad (5.2)$$

where $n$ is the element number, $a_n$ is the apodization coefficient, $\psi_a^{(n)}$ and $\psi_b^{(n)}$ are the phase delay functions, and $H(r, \omega)$ is the spatial impulse function in the frequency-domain of a rectangular element. The function $H(r, \omega)$ is given by Eq. (4.6), where the spatial impulse function $h(r, t)$ should be calculated for a rectangular piston. The exact expressions for $h(r, t)$ of a rectangular piston is given in Ref. [89]. The vectors $\mathbf{r}_a^{(n)} = \left(x_a^{(n)}, y_a^{(n)}, 0\right)$ and $\mathbf{r}_b^{(n)} = \left(x_b^{(n)}, y_b^{(n)}, 0\right)$ are the center of the $n$th-element in the arrays $a$ and $b$, respectively. Tables 5.1 and 5.2 show the position of the elements for all transducers discussed here. For a focal distance $r_0$ and a steering angle $\theta_0$, the phase delay functions are given by

$$\psi_a^{(n)} = \frac{\omega_a}{c_0} \left(r_0 - \sqrt{r_0^2 + x_a^{(n)} - 2r_0x_a^{(n)} \sin \theta_0}\right), \quad (5.3)$$

$$\psi_b^{(n)} = \frac{\omega_b}{c_0} \left(r_0 - \sqrt{r_0^2 + x_b^{(n)} - 2r_0x_b^{(n)} \sin \theta_0}\right). \quad (5.4)$$
Figure 5.4: Dual parallel linear array transducers with $N = 8$. Each transducer has two linear arrays $a$ and $b$. (a) symmetrically separated, (b) centered, and (c) interlaced. Both arrays have horizontal pitch $d$ and vertical pitch $d_v$. The array elements have width $w$ and height $l$.

We are interested in analyzing how the PSF in (2.54) changes for six different geometric arrangements of the arrays. The six transducer arrangements are divided into two categories: dual inline linear arrays and dual parallel linear arrays. In Fig. 5.3, we have the dual inline linear array transducers: (a) symmetrically separated, (b) centered, and (c) interlaced. Fig. 5.4 shows the dual parallel linear array transducers separated by a distance $d_v$: (a) symmetrically separated, (b) centered, and (c) interlaced. The two arrays in each transducer have the same characteristics except for the driving signal.

5.3 Computational method

A computer program, acoustic field simulator (AFS) written in C programming language, was developed to calculate the system PSF based on the linear array transducers. In Fig. 5.5, we have a block diagram of the AFS. The program was executed in a PC-Pentium with Linux operating system. A description of each module is given as follows:

+ Name: afs_main.c
  Description: This module is responsible to call input/output (I/O) routines that have the information of the array to be simulated. The module sends the information to afs_mesh and afs_beamforming.c.

Name: afs_io.c
Table 5.1: The position of the elements in the inline linear arrays.

In all configurations $y_a^{(n)} = y_b^{(n)} = 0$ and $n = 1, 2, \ldots, N$
(except in the centered array).

<table>
<thead>
<tr>
<th>Transducer</th>
<th>Element position</th>
</tr>
</thead>
<tbody>
<tr>
<td>separated</td>
<td>$x_a^{(n)} = (n - N - 1/2) d$</td>
</tr>
<tr>
<td></td>
<td>$x_b^{(n)} = (n - 1/2) d$</td>
</tr>
<tr>
<td>centered</td>
<td>$x_a^{(n)} = \begin{cases} (n - N - 1/2) d, &amp; n = 1, 2, \ldots, N/2 \ (n + N/2 - 1/2) d, &amp; n = N/2 + 1, N/2 + 2, \ldots, N \end{cases}$</td>
</tr>
<tr>
<td></td>
<td>$x_b^{(n)} = (n - N/2 - 1/2) d$</td>
</tr>
<tr>
<td>interlaced</td>
<td>$x_a^{(n)} = (n - N/2 - 3/4) d + w$</td>
</tr>
<tr>
<td></td>
<td>$x_b^{(n)} = (n - N/2 - 5/4) d + w$</td>
</tr>
</tbody>
</table>

Table 5.2: The position of the elements in the parallel linear arrays.

In all configurations $x_a^{(n)} = x_b^{(n)} = (n - N - 1/2) d$ and $n = 1, 2, \ldots, N$
(except in the centered array).

<table>
<thead>
<tr>
<th>Transducer</th>
<th>Element position</th>
</tr>
</thead>
<tbody>
<tr>
<td>separated</td>
<td>$y_a^{(n)} = (l + d_v)/2$</td>
</tr>
<tr>
<td></td>
<td>$y_b^{(n)} = -(l + d_v)/2$</td>
</tr>
<tr>
<td>centered</td>
<td>$y_a^{(n)} = \begin{cases} -(l + d_v)/2, &amp; n = 1, 2, \ldots, N/4, 3N/4 + 1, \ldots, N \ (l + d_v)/2, &amp; n = N/4 + 1, N/4 + 2, \ldots, 3N/4 \end{cases}$</td>
</tr>
<tr>
<td></td>
<td>$y_b^{(n)} = \begin{cases} (l + d_v)/2, &amp; n = 1, 2, \ldots, N/4, 3N/4 + 1, \ldots, N \ -(l + d_v)/2, &amp; n = N/4 + 1, N/4 + 2, \ldots, 3N/4 \end{cases}$</td>
</tr>
<tr>
<td>interlaced</td>
<td>$y_a^{(n)} = (-1)^n (l + d_v)/2$</td>
</tr>
<tr>
<td></td>
<td>$y_b^{(n)} = (-1)^{n-1} (l + d_v)/2$</td>
</tr>
</tbody>
</table>

Description: This is the module responsible for all I/O operations in the AFS.

Name: afs_mesh.c

Description: In this module, the lattice (3D-spatial points) representing the propagating medium is generated. Also, the mesh grid representing the simulated array is produced.

Name: afs_geometry.c

Description: All geometric operations of the AFS such rotation and distance determination of objects takes place in this module.

Name: afs_math.c

Description: The math operations such as matrix product and sum is realized in this module.

Name: afs_beamforming.c
Figure 5.5: Diagram of the modules of the acoustic field simulator program

Description: This module separates the information of each linear array in the transducer and pass them to afs_impulse.

Name: afs_time.c
Description: The delays of the driving signals applied to the array elements for focusing purpose are calculated in this module.

Name: afs_apodization.c
Description: Eight different apodization functions for the array elements are implemented in this module.

Name: afs_impulse.c
Description: The functions to calculate the spatial impulse function $h(r,t)$ of a rectangular piston are in this module. The code of the main function of the module can be seen in Appendix A. Basically, this module calculates the spatial impulse functions in (5.1) and (5.2) by using Eq. (4.6) with $h(r,t)$ given by Ref. [89]. Eq. (4.6) is numerically evaluated using the trapezoidal rule. Figure 5.6 shows the profiles of the spatial impulse function $h(r,t)/c_0$ of a rectangular pistons of dimension $a$ and $b$. The results shown are in complete agreement with Ref. [89].

To simulate acoustic cylindrical lens each array element in the AFS can be divided in $n$ rectangular sub-elements in elevation. A time delay is applied on the sub-elements to focus the ultrasound beams in elevation at a specific distance in range.
5.4 Results

Table 5.3 summarizes the transducer parameters used in the computational simulation. To compare the PSF of the transducers, we fixed the total aperture size to \((25.6 \times 12)\) mm laterally and in elevation for all transducers. The speed of sound was set to \(c_0 = 1480\) m/s. Figure 5.7 shows the planes (azimuth and elevation) where the PSF was calculated. The focal point of the transducer is specified by the polar coordinates \((r_0, \theta_0)\).

5.4.1 Dynamic radiation force

Before calculating the components of the dynamic radiation force, it is worthwhile to investigate if the conditions (3.65) and (3.66) hold for ultrasound waves radiated by linear arrays. We also need to know what is the largest intersecting angle \(\alpha\) between the incident velocity fields of each ultrasound wave generated by the linear array transducers discussed here. This is important because the dynamic radiation force may have a contribution from Eq. (3.62). To estimate \(\alpha\), let us assume that the waves propagate as rays in the focal zone of the transducer. In this case, the velocity fields produced by the inline separated transducer have the largest intersecting angle among all transducers. In fact, the other transducers have this angle practically equal to zero. One can show that for the inline
Figure 5.7: Planes where the PSF is calculated. The elevation plane is perpendicular to range. The center of the planes are specified by the transducer focus point \((r_0, \theta_0)\).

Table 5.3: Parameters used in the computer simulations.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of elements ((N))</td>
<td>64 (inline) and 128 (parallel)</td>
</tr>
<tr>
<td>element width ((w))</td>
<td>0.15 mm</td>
</tr>
<tr>
<td>element height ((l))</td>
<td>12 mm (inline) and 6 mm (parallel)</td>
</tr>
<tr>
<td>pitch ((d))</td>
<td>0.2 mm (0.4 mm for the inline interlaced transducer)</td>
</tr>
<tr>
<td>vertical pitch ((d_v))</td>
<td>0</td>
</tr>
<tr>
<td>lateral aperture width ((D))</td>
<td>25.6 mm</td>
</tr>
<tr>
<td>aperture width in elevation ((L))</td>
<td>12 mm</td>
</tr>
<tr>
<td>focus in elevation</td>
<td>50 mm</td>
</tr>
<tr>
<td>ultrasound frequency ((f_0))</td>
<td>3 MHz</td>
</tr>
<tr>
<td>difference frequency ((\Delta f))</td>
<td>10 kHz</td>
</tr>
</tbody>
</table>

separate transducer the angle between the incident velocity fields is given by

\[
\alpha = \frac{2r_0^2 - D^2}{\sqrt{\left(r_0^2 + D^2/4\right)^2 - D^2r_0^2\sin\theta_0}},
\]

where \(D\) is the lateral aperture width. For \(\theta_0 = 0\) we have \(\alpha = 14.4^\circ\). As we increase \(\theta_0\) the angle \(\alpha\) decreases. Therefore, we may neglect the contribution of Eq. (3.62) to the dynamic radiation force.

Figure 5.8 shows plots of \(\frac{1}{\text{max } \phi^{(a)}} \frac{\partial \phi^{(a)}}{\partial z}\) and \(\frac{1}{\text{max } \phi^{(b)}} \frac{\partial \phi^{(b)}}{\partial z}\). The amplitude of both plots is 12.5 mm\(^{-1}\) which corresponds to the value \(\frac{\omega_0 c_0}{c_0}\) used in the simulation. The amplitudes of the ultrasound waves \(a\) and \(b\) are plotted in Figure 5.10. The wave pattern resembles collimated plane waves. The radiation force in the \(xz\)-plane can be seen in Figure 5.10.c. In the focal zone, the dynamic radiation force points to the axial direction. Therefore, the conditions (3.65) and (3.66) are applicable for the inline separated
transducer and the main contribution for the dynamic radiation force comes from the axial component of the force.

The axial and transverse components of the dynamic radiation force produced by a inline separated transducer without steering are shown in the Figures 5.9 and 5.11. The transducer is focused at \( r_0 = 50 \text{ mm} \) in range. The magnitude of the transverse force is only 3.4\% \((-29.2 \text{ dB})\) of the peak of the axial force. Figure 5.12 shows the axial force when the ultrasound beam is steered by 30\(^\circ\). The magnitude of the transverse force remains 3.4\% of the peak of the axial force. However, the transverse vector field is asymmetric in the mainlobe. Because of the small magnitude of the transverse radiation force, we can continue the analysis of the PSF of linear array transducers based only in the axial component of the radiation force.

### 5.4.2 Point-spread function

Figure 5.13 shows the PSF of the linear array transducers evaluated along the arc \(-30^\circ < \theta < 30^\circ\) at the focal distance \( r_0 = 50 \text{ mm} \). The inline separated transducer has the widest main lobe width (2.4 mm) at \(-12 \text{ dB}\) from its peak. The rest of the transducers have the main lobe width of about 1.2 mm. All transducers have sidelobes under \(-26 \text{ dB}\), except the inline centered transducer whose sidelobes are just under \(-7.4 \text{ dB}\).

Figure 5.14 shows the PSF profile of all transducers in range. At \(-12 \text{ dB}\) from the peak, the
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Figure 5.9: Axial component of the dynamic radiation force on the focal plane of the inline separated array. The arrows represent the transverse radiation force vector.

PSF width of all transducers is about 18 mm, except for the inline centered transducer whose width is 29.5 mm. In all transducers the sidelobes are below $-14.0$ dB.

Figure 5.15 shows the PSF profiles of all transducers in elevation. At $-12$ dB from the peak, the PSF of the inline transducers has a width of about 2.6 mm with sidelobes below $-28.8$ dB. The PSF of the parallel separated and centered transducers has width about 5 mm at $-12$ dB and sidelobes below $-26.8$ dB. The PSF width of the parallel interlaced transducer is 2.6 mm at $-12$ dB and sidelobes are below $-27.0$ dB.

5.4.3 Grating lobes and sidelobes

Linear arrays produce grating lobes when the array pitch is $d > \lambda/2$, where $\lambda$ is the ultrasound wavelength. In our simulations, the ultrasound wavelength is $\lambda \simeq 0.5$ mm. The inline interlaced transducer exhibits grating lobes because the array pitch is $d = 0.4$ mm. In fact, each array individually produces grating lobes that overlap in space. Hence, the product of the two beams in Eq. (2.54) also has grating lobes. Figure 5.16 shows grating lobes in the PSF of the interlaced transducers. When the beam is steered from $0^\circ$ to $-40^\circ$ the grating lobe level increases from $-36.4$ dB to $-4.4$ dB. The other transducers do not exhibit grating lobes because $d < \lambda/2$.

The sidelobes present in the vibro-acoustography PSF come from the overlap of the sidelobes of ultrasound beams $a$ and $b$. By reducing the sidelobes of the beam in each array, the sidelobes in the system PSF is reduced. Figure 5.17 shows the PSF of the inline and parallel interlaced trans-
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Figure 5.10: Ultrasound fields produced by the inline separated array on the $xz$-plane. (a) Ultrasound wave $a$. (b) Ultrasound wave $b$. (c) Radiation force distribution. The arrows represent the force vector field in the plane.

Producers, in which each individual array was apodized by the Bartlett’s function

$$a_n = 1 - \left| \frac{2x^{(n)}}{D} \right|,$$  \hspace{1cm} (5.6)

where $x^{(n)}$ is the center of the $n^{th}$-array element and $D = 25.6$ mm is the transducer lateral aperture. In both transducer after the apodization, the sidelobe levels of the system PSF are as low as $-36.0$ dB and main lobe is $5$ dB lower than the non-apodized PSF. Furthermore, the main lobe width at $-12$ dB becomes $12\%$ wider after the apodization.

5.4.4 Effects of beam steering and focusing

The shape of the resolution cell is subject to distortions as the total ultrasound beam is steered and focused at different points in the medium. This effect is shown in the separated and interlaced transducers. We define the relative broadening of the resolution cell as $\Delta \eta \equiv (1 - \eta_1/\eta_2) \times 100\%$, where $\eta_i$ is the cell width along range, azimuth, or elevation. The ultrasound beam is focused at the position $i = 1, 2$, specified in polar coordinates by $(r_1, \theta_1)$ and $(r_2, \theta_2)$, respectively. Because the interlaced transducers present the best performance according to Figs. 5.13, 5.14, and 5.15, we shall focus on these transducers comparing them mainly with the inline separated transducer.
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Figure 5.11: Transverse component of the dynamic radiation force on the focal plane of inline separated transducer. The arrows represent the transverse radiation force vector. The magnitude is normalized by the peak value of the axial component of the force.

Figure 5.18 shows contour plots of the system PSF in the azimuth plane. The positions 1 and 2 are (30 mm, 0°) and (70 mm, 0°). The relative broadening of the resolution cell along the range is $\Delta \eta = 78.2\%$. Figure 5.19 shows contour plots of the system PSF in the elevation plane. In azimuth, the relative broadening of all transducers is $\Delta \eta = 54.3\%$, except the inline separated for which $\Delta \eta = 55.4\%$. In elevation, all transducers have have $\Delta \eta = -11.1\%$, except the parallel separated transducer, which has $\Delta \eta = 23.5\%$.

Figure 5.20 shows contour plots of the system PSF in the azimuth plane with the beam steered to $\theta_0 = 30^\circ$ for the inline separated transducer and the interlaced transducers. In this case, the positions 1 and 2 are, respectively, (30 mm, 30°) and (70 mm, 30°). Besides the distortion, the resolution cells are no longer symmetric in range. The relative broadening in range for all transducers is $\Delta \eta = 68.5\%$.

In Figure 5.21, we have contour plots of the system PSF, with the beam steered at $\theta = 30^\circ$, in the elevation plane rotated by 30° around the y-axis (see Figure 5.7) for the inline separated transducer and the interlaced transducers. In azimuth, the parallel transducers have $\Delta \eta = 53.1\%$, while the inline transducers have $\Delta \eta = 59.0\%$. In elevation, all transducers have $\Delta \eta = -14.2\%$. Consider the positions 1 and 2 as (50 mm, 0°) and (50 mm, 30°). In this case, for all transducers $\Delta \eta = 23.0\%$ in range. In azimuth, all transducers have $\Delta \eta = 18.4\%$, except the inline separated transducer that has $\Delta \eta = 7.7\%$. In elevation, all transducers have $\Delta \eta = 6.5\%$. 
5.4.5 Numerical error

The main source of error on the simulating program is the numerical evaluation of Eq. (4.6) due to the discontinuities in time of $h(r,t)$. The mean error in numerical calculation of pressure fields using the results of Ref. [89] and the fast Fourier transform is less than 10% for sampling frequencies above 500 MHz [90]. In our simulations, the sampling frequencies were above this limit. Hence, it is expected that our mean error to be less than 10%.

5.5 Real-time systems

The design of a real-time vibro-acoustography system should take into account two intrinsic characteristics of the system image synthesis. First, vibro-acoustography produces images based on acoustic emission in steady-state. The vibrations caused by the dynamic radiation force on a tissue-like medium may need few cycles at the frequency $\Delta \omega$ to reach the steady-state. Second, the time-of-flight of the acoustic emission from the scanned point to the detector may take few microseconds. The longest time-of-flight of the acoustic emission and the time interval required to reach the vibration steady-state determines how many frames per second the system can exhibit. If $n_p$ pixels are displayed per frame and $n_c$ is the number of cycles needed for the acoustic emission to reach the steady-state,
the number of frames per second of the system is

$$n_f = \left[ n_p \left( \frac{2\pi n_c}{\Delta \omega} + \frac{d_{\text{max}}}{c_0} \right) \right]^{-1},$$

(5.7)

where $d_{\text{max}}$ is the distance between the farthest scanned point in the medium and the detector. The quantity $d_{\text{max}}/c_0$ is the longest time-of-flight of the acoustic field emitted to the detector.

Let us consider a system based on the interlaced parallel transducer specified by Table 5.3 producing a B-scan image centered at 50 mm away of the transducer with (20 × 20) mm of area. In polar coordinates this area corresponds to $r_0 = 40$ mm to 60 mm and $\theta_0 = -11.3^\circ$ to 11.3°. For a scanning area of with $n_p = 1600$ pixels, $n_c = 5$ cycles, and $d_{\text{max}} = 50$ mm, the minimum difference-frequency to produce one frame per second, according to Eq. (5.7), is 8 kHz.

### 5.6 Summary

The linear array beamforming for vibro-acoustography was studied in this chapter. It was shown that the quasi-plane wave model for the radiation force is still a good approximation for the linear array beamforming. The variation of the system resolution cell as the focal point is changed spatially was shown through several computational simulations. Based on the results, we concluded that vibro-acoustography systems based on linear arrays for clinical applications are feasible.
Figure 5.14: PSF evaluated in range from 10 mm to 90 mm. Inline transducers: (a) separated, (b) centered, and (c) interlaced. Parallel transducers: (d) separated, (e) centered, and (f) interlaced.

Figure 5.15: PSF evaluated in elevation. Inline transducers: (a) separated, (b) centered, and (c) interlaced. Parallel transducers: (d) separated, (e) centered, and (f) interlaced.
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Figure 5.16: Grating lobes produced by the inline interlaced transducer. (a) Grating lobes are under $-36.4\text{ dB}$ at $\theta = \pm 90^\circ$ when the beam is not steered. (b) Grating lobe at $\theta = 40^\circ$ with amplitude of $-4.4\text{ dB}$ when the beam is at $\theta = -40^\circ$.

Figure 5.17: Effects of apodization with the Bartlett’s function in the system PSF. (a) inline interlaced transducer. (b) parallel interlaced transducer.
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Figure 5.18: Contour plots of the system PSF in the azimuth plane. The beam is focused at $r_0 = 30$, $50$, and $70$ mm without steering. (a)-(c) Inline separated transducer. (d)-(f) Inline interlaced transducer. (g)-(i) Parallel separated transducer. (j)-(l) Parallel interlaced transducer. Legend: solid line $-30$ dB, thick line $-12$ dB, and dotted line $-6$ dB.

Figure 5.19: Contour plots of the system PSF in the elevation plane. The beam is focused at $r_0 = 30$, $50$, and $70$ mm without steering. (a-c) Inline separated transducer. (d-f) Inline interlaced transducer. (g-i) Parallel separated transducer. (j-l) Parallel interlaced transducer. Legend: solid line $-30$ dB, thick line $-12$ dB, and dotted line $-6$ dB.
Figure 5.20: Contour plots of the system PSF in the azimuth plane. The beam is focused at $r_0 = 30$, 50, and 70 mm with steering angle $\theta_0 = 30^\circ$. (a-c) Inline separated transducer. (d-f) Inline interlaced transducer. (g-i) Parallel interlaced transducer. Legend: solid line $-30$ dB, thick line $-12$ dB, and dotted line $-6$ dB.

Figure 5.21: Contour plots of the system PSF in the elevation plane rotated by $30^\circ$ around y-axis. The beam is focused at $r_0 = 30$, 50, and 70 mm with steering angle $\theta_0 = 30^\circ$. (a-c) Inline separated transducer. (d-f) Inline interlaced transducer. (g-i) Parallel interlaced transducer. Legend: solid line $-30$ dB, thick line $-12$ dB, and dotted line $-6$ dB.
Chapter 6

Summary and conclusion

6.1 Summary

The main goal of this dissertation was to present a study of beamforming and imaging synthesis in vibro-acoustography systems. A generalization of the collimated quasi-plane wave model was proposed based on the Brillouin radiation-stress tensor. The point-spread function (PSF) of the system, which depends on the response of a point-target to the dynamic radiation force, was obtained. The axial and transverse components of the radiation force exerted by any ultrasound modulated beam can be calculated from the theory presented here. The radiation force exerted by a traveling ultrasound wave on a point-target is proportional to the ultrasound intensity vector. The vibro-acoustography resolution cell may have a complicated pattern of vibration. However, the main contribution for the dynamic radiation force comes from the axial component of the force. The transverse force corresponds to 3-11% of the axial component for different transducers. Even though this represents a small amount of the total force, it may produce, combined with sidelobes, some sort of image artifacts. For example, the edge brightness effect observed in some artery images (see Figure 1.3). However, more studies need to be continued in this subject. Theoretical expression for the PSF of an specific transducer based on the three-dimensional radiation force vector can be used as a starting point for deconvolution of vibro-acoustographic images.

An analytic expression of the acoustic potential of a sector transducer is obtained from the Rayleigh integral. Theoretically, sidelobes in the PSF of the 8-element sector array transducer are under $-15.8$ dB in eight spots circularly distributed and separated by $22.5^\circ$. Simulations show that for $N = 16$ sidelobe levels become less than $-34$ dB. This happens because by increasing the number of sectors we get closer to AM-single beam mode whose sidelobes are under $-35$ dB. Sidelobes in the PSF of a confocal transducer is about $-18.2$ dB.
Special attention has been given to linear array transducers because these transducers can be used for clinical application of vibro-acoustography system. Currently, vibro-acoustography systems are based on confocal or sector transducers. The feasibility of six different linear array configurations for vibro-acoustography was analyzed. The inline interlaced transducer with $N = 64$ and the parallel interlaced transducer with $N = 128$ presented the best resolution cells. With the beam focused at 50 mm without steering the resolution cell of these transducers is $(1.2 \times 2.6 \times 18)$ mm in azimuth, elevation, and range. Better resolution in range can be achieved by increasing the number of array elements. The sidelobes in the interlaced transducers were reduced to less than $-37.6$ dB through the apodization technique. Variations in the resolution cell due to focusing and steering may reduce image quality. The relative broadening of the resolution cell has more variation in range $\Delta \eta = 78.2\%$, as the beam is focused from 30 mm to 70 mm. When the beam is steered from $0^\circ$ to $30^\circ$ at 50 mm, the maximum variation is $\Delta \eta = 23.0\%$ in range. These variations can be minimized if the transducer aperture varies as a function of the focus position. If the focus point is far away from the transducer, more active array elements are needed to keep the same resolution as the focus point is closer to the transducer.

The PSF of the linear array transducers have better resolution transversally than in depth. This indicates that the linear array transducers analyzed here would produce better images in C-scan compared to B-scan. Nevertheless, C-scan images require beam steering and focusing in elevation. Beam steering in elevation can be accomplished by mechanically steering the ultrasound beam or by using 2D-linear arrays. The latter solution involves certain complexity in probe and hardware manufacturing due to the large number of array elements. The theory and simulation presented in this work can be adapted without major modifications to study 2D-linear array beamforming for vibro-acoustography. The inline and the parallel interlaced transducers presented the best resolution cells among the studied transducers. Moreover, the path of the two ultrasound beam produced by the arrays in these transducer are practically the same. Having a common path for both ultrasound beams may reduce the error of focusing the beams due to inhomogeneities or variations on the speed of sound in tissue. The inline interlaced transducer could be used as a switched linear array in linear scanning mode. For this configuration, the grating lobes are under $-36$ dB and sidelobes can be reduced to less than $-36$ dB by applying apodization. The parallel interlaced transducer could be used as a phased linear array because the resulting beam can be steered without grating lobes.

### 6.2 Conclusion

We presented a theory to describe the spatial resolution cell of a vibro-acoustography system. This theory can be only applied to ideal fluids. Despite this limitation, the theory provides a
useful framework for vibro-acoustography image formation based on linear acoustics. This includes the analysis of the system PSF for any ultrasound beam and a description of the dynamics of the system resolution cell.

The radiation force produced by confocal and sector transducers were validated experimentally. The transverse and axial components of the radiation force were measured for a confocal transducer using a steel wire as a target. For the sector transducer only the axial component was measured. The results are in good agreement with the theoretical predictions. No experimental data for linear array transducers are available up until now. We are still using the computational simulation for guidance in the design of systems based on linear arrays.

### 6.3 Future work

Ultrasound vibro-acoustography is a very young imaging modality. The theory behind the scenes of vibro-acoustography is still not quite understood. The study of vibro-acoustography in different propagating mediums such as viscous fluids, viscoelastic materials, or solids, consists a set of new problems. Effects of nonlinearity of the propagating medium could also be explored. Optimization of vibro-acoustography beamforming based on intrinsic characteristics of system can be studied. Specially using random linear array transducers. Finally, validation of the results for linear arrays is a natural path of this work.
Appendix A

Computer code for evaluation of the spatial impulse function

We start from the spatial impulse function definition

\[ h(\mathbf{r}, t) = \int_{\Omega} \frac{\delta(t - |\mathbf{r} - \mathbf{r}'|/c_0)}{2\pi|\mathbf{r} - \mathbf{r}'|} dS(\mathbf{r}'). \]  
\[ \text{(A.1)} \]

This a statement of Huygens principle that the field is found by summing the radiated spherical waves from all parts of the piston surface. This can be reformulated, by using acoustic reciprocity, as finding the part of the spherical wave emanating from the field point that intersects the piston surface. Hence the task is to project the field point onto the plane coinciding with the piston surface, and then find the intersection of the projected spherical wave (a circle) with the active surface \( \Omega \). Writing this integral in polar coordinates we have

\[ h(\mathbf{r}, t) = \int_{d_1}^{d_2} \int_{\theta_1}^{\theta_2} \frac{\delta(t - R/c_0)}{2\pi R} r' dr' d\theta', \]

where \( r' \) is the radius of the projected circle and \( R = |\mathbf{r} - \mathbf{r}'| \) is the distance from the field point to the piston surface given by \( R^2 = r^2 + z_d^2 \). Here \( z_d \) is the distance of the point to the \( xy \)-plane (baffled plane). The projected distances \( d_1 \) and \( d_2 \) are the closest and the farthest distances from the projected point to the edges of \( \Omega \), and \( \theta_1 \) and \( \theta_2 \) are the corresponding angles for a given time. Introducing the transformation \( RdR = rdr \) we have

\[ h(\mathbf{r}, t) = \int_{R_1}^{R_2} \int_{\theta_1}^{\theta_2} \delta(t - R/c_0) dRd\theta', \]  
\[ \text{(A.2)} \]
Appendix A: Computer code for evaluation of the spatial impulse function

where the quantities $R_1$ and $R_2$ denote the closest and farthest edges away to the field point. Using the substitution $t' = R/c_0$ yields

$$ h(r,t) = \frac{c_0}{2\pi} \int_{\theta_1}^{\theta_2} \int_{t_1}^{t_2} \delta(t-t') dt'd\theta'. \quad \text{(A.3)} $$

For a given time instance the contribution along the arc is constant and the integral gives

$$ h(r,t) = \frac{c_0}{2\pi} (\theta_2 - \theta_1). \quad \text{(A.4)} $$

The angles $\theta_1$ and $\theta_2$ are determined by the intersection of the aperture and the projected spherical wave. Thus, the spatial impulse response can be determined by keeping track of the intersections as a function of time. Below we have a computer code in C programming language which calculates the angles $\theta_1$ and $\theta_2$ based on the results of Ref. [89].

**Program code in C**

```c
/* afs_impulse_rect.c - Calculate the spatial impulse response of a
* rectangular aperture in a point r
* Usage: void afs_impulse_rect( r, a, b, c, tI, fs )
* r - point in the 3D-space
* a,b - width and height
* c - sound speed
* tI - initial time
* fs - sampling frequency
* Glauber T. Silva
* February, 2002.
*/
void afs_impulse_rect(
    Point_t r,
    double a,
    double b,
    double c,
    double* tI_ptr,
    double fs
)
{
    unsigned int l, ns;
    double d1, d2, d3, d4, d_f1,d_f2, d_f3, d_f4;
    double d_s1, d_s2, d_s3, d_s4;
    double t, t0, ta, tb, tc, td, tmin, tmax, t_ini;
    double tsi, ts2;
    double sigma, dt, z_2, x, y, z, cl;

    x = fabs(r.x);
```
Appendix A: Computer code for evaluation of the spatial impulse function

```c
y = fabs(r.y);
z = fabs(r.z);

/* a>b */
if ( a>b )
{
    SWAP(x,y);
    SWAP(a,b);
}

a *= .5;
b *= .5;
c1 = 1./(2.*PI);
ns = 0;

/* Distance to the field point to the straight lines to rect. lines */
d1 = x - a;
d2 = y - b;
d3 = x + a;
d4 = y + b;
d_f1 = ABS(d1);
d_f2 = ABS(d2);
d_f3 = ABS(d3);
d_f4 = ABS(d4);
d_s1 = (double) SGN(d1);
d_s2 = (double) SGN(d2);
d_s3 = (double) SGN(d3);
d_s4 = (double) SGN(d4);

/* Sampling interval */
dt = 1./fs;

/* Transit times from the vertices to a field point */
z_2 = z*z;
ta = sqrt(d1*d1 + d2*d2 + z_2) / c;
tb = sqrt(d2*d2 + d3*d3 + z_2) / c;
tc = sqrt(d1*d1 + d4*d4 + z_2) / c;
td = sqrt(d3*d3 + d4*d4 + z_2) / c;
l = 0;
/* Region I: x>=a, y>=b */
if (x>=a && y>=b)
{
    /* Calculate the number of sampled points */
    ns = ceil((td - ta)*fs);
    /* Implement equations (11) and (12) of the reference */
    t_ini = ta;
t = t_ini;
tmin = MIN(tb,tc);
tmax = MAX(tb,tc);
    while (t>=ta && t<=tmin)
    {
        sigma = sqrt(c*c*t*t - z_2);
        *(H ->value+l) = (sigma>0.) ? c1 * (.5*PI - ALPHA(d1/sigma)
```

Appendix A: Computer code for evaluation of the spatial impulse function

```c
-ALPHA(d2/sigma) : 0.;

l++;  
t += dt;
}
while (t>tmin & t<=tmax)
{
    sigma = sqrt(c*c*t*t - z_2);
    if (tmin==tb)
        *(H ->value+l) = c1 * ( ALPHA(d3/sigma) - ALPHA(d1/sigma) );
    else
        *(H ->value+l) = c1 * ( ALPHA(d4/sigma) - ALPHA(d2/sigma) );
l++;  
t += dt;
}
while (t>tmax & t<=td)
{
    sigma = sqrt(c*c*t*t - z_2);
    *(H ->value+l) = c1 * ( -.5*PI + ALPHA(d3/sigma) + ALPHA(d4/sigma) );
l++;  
t += dt;
}
else
/* Region II: x<=a, y>=b */
if (x<=a & y>=b)
{
    ts2 = sqrt(d2*d2 + z_2)/c;

    /* Calculate the number of sampled points */
    ns = ceil((td - ts2)*fs);  
t_ini = ts2;  
t = t_ini;
while ( t>=ts2 & t<ta )
{
    sigma = sqrt(c*c*t*t - z_2);
    *(H ->value+l) = (sigma>0.) ? c1 * ( PI - 2.*ALPHA(d2/sigma) ) : 0.;
l++;  
t += dt;
}
while (t>ta & t<=tb)
{
    sigma = sqrt(c*c*t*t - z_2);
    *(H ->value+l) = c1 * ( .5*PI - ALPHA(d1/sigma) - ALPHA(d2/sigma) );
l++;  
t += dt;
}
while (t>tb & t<=tc)
{
    sigma = sqrt(c*c*t*t - z_2);
    *(H ->value+l) = c1 * ( -PI - ALPHA(d1/sigma) + ALPHA(d3/sigma)
    + 2.*d_s4*ALPHA(d_f4/sigma) );
l++;  
t += dt;
}

```

t += dt;
} while (t>tc && t<=td)
{
    sigma = sqrt(c*c*t*t - z_2);
    *(H ->value+l) = cl * ( -.5*PI + ALPHA(d3/sigma) + ALPHA(d4/sigma) );
    l++;
    t += dt;
}
else
/* Region III: x>=a, y<=b */
if (x>=a && y<=b)
{
    tsl = sqrt(d1*d1 + z_2)/c;
    /* Calculate the number of sampled points */
    ns = ceil((td - tsl)*fs);
    t_ini = tsl;
    t = t_ini;
    tmin = MIN(tb,tc);
    tmax = MAX(tb,tc);
    while ( t>=tsl && t<ta )
    {
        sigma = sqrt(c*c*t*t - z_2);
        *(H ->value+l) = (sigma>0.) ? cl * ( 2.*(d_s3)*ALPHA(d_f3/sigma)
            - 2.*ALPHA(d1/sigma) ) : 0.;
        l++;
        t += dt;
    }
    while (t>=ta && t<=tmin)
    {
        sigma = sqrt(c*c*t*t - z_2);
        *(H ->value+l) = cl * ( -.5*PI - ALPHA(d1/sigma)
            - 2.*ALPHA(d2/sigma) + 2.*d_s3 * ALPHA(d_f3/sigma) );
        l++;
        t += dt;
    }
    while (t>tmin && t<=tmax)
    {
        sigma = sqrt(c*c*t*t - z_2);
        if (tmin==tb)
            *(H ->value+l) = cl * ( -ALPHA(d1/sigma) + ALPHA(d3/sigma));
        else
            *(H ->value+l) = cl * ( -PI - ALPHA(d2/sigma)
                + ALPHA(d4/sigma)
                + 2.* d_s3 * ALPHA(d_f3/sigma) );
        l++;
        t += dt;
    }
    while (t>tmax && t<=td)
    {
        sigma = sqrt(c*c*t*t - z_2);
        *(H ->value+l) = cl * ( -.5*PI + ALPHA(d3/sigma)

Appendix A: Computer code for evaluation of the spatial impulse function

```c
int l;
double t;
for (t = t0; t < ta; t += dt)
{
    l++;
    t += dt;
}
```

```c
else
/* Region IV: x<=a, y<=b */
if (x<=a && y<=b)
{
    t0 = z/c;
    /* Calculate the number of sampled points */
    ns = ceil((td - t0)*fs);

    t_init = t0;
    t = t_init;
    tmin = MIN(tb,tc);
    tmax = MAX(tb,tc);
    while ( t>=t0 && t<ta )
    {
        sigma = sqrt(c*c*t*t - z_2);
        *(H ->value+l) = (sigma>0.) ? 2. * cl * { - PI
            - d_s1*ALPHA(d_f1/sigma)
            - d_s2*ALPHA(d_f2/sigma)
            + d_s3*ALPHA(d_f3/sigma)
            + d_s4*ALPHA(d_f4/sigma) }
            : 0.;
        l++;
        t += dt;
    }

    while ( t>=ta && t<tmin )
    {
        sigma = sqrt(c*c*t*t - z_2);
        *(H ->value+l) = cl * { -1.5*PI - ALPHA(d1/sigma)
            - ALPHA(d2/sigma)
            + 2.*d_s3*ALPHA(d_f3/sigma)
            + 2.*d_s4*ALPHA(d_f4/sigma) }
            ;
        l++;
        t += dt;
    }

    while ( t>=tmin && t<=tmax )
    {
        sigma = sqrt(c*c*t*t - z_2);
        if (tmin==tb)
            *(H ->value+l) = cl * { -PI
                - ALPHA(d1/sigma) + ALPHA(d3/sigma)
                + 2.*d_s4*ALPHA(d_f4/sigma) }
            ;
        else
            *(H ->value+l) = cl * { -PI - ALPHA(d2/sigma)
                + ALPHA(d4/sigma)
                + 2.*d_s3*ALPHA(d_f3/sigma) }
            ;
        l++;
        t += dt;
    }
```
while (t>tmax && t<=td) {
    sigma = sqrt(c*c*t*t - z_2);
    *(H ->value+l) = c1 * ( -.5*PI + ALPHA(d3/sigma) + ALPHA(d4/sigma) );
    l++;
    t += dt;
}
*tI_ptr = t_ini;
H->cols = ns;
}
Bibliography


